MSc In Quantum Fields and Fundamental Forces

Imperial College London

Department of Physics

A Gentle Introduction to Topological Quantum Field Theory and Mirror Symmetry

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Abstract

This project presents a review of the work originally pioneered by E. Witten on topological quantum field theories and how mirror symmetry arises in them. A discussion of differential geometry leads into a rigorous, in depth look at what topological quantum field theories are. Both the mathematical and physical sides are discussed. The second half of this project reviews how the $\mathcal{N} = (2, 2)$ supersymmetry nonlinear σ-model is topological in nature and its connection to mirror symmetry.

Acknowledgements

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Chapter 1

Introduction

We are all familiar with the sentiment that physics is the product of mathematics. It is hardly ever heard the other way around, that mathematics is the product of physics. As we will see, the statement actually can be swapped; mathematical questions can be answered by physical methods and techniques. Let us motivated this further.

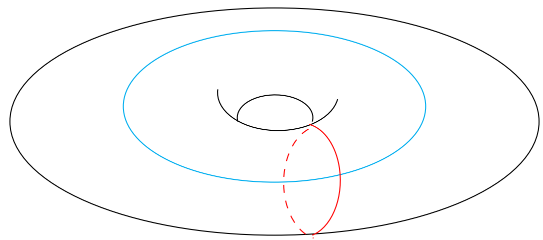
1.1 Setting the Stage

One of the main focuses of mathematics is the classification and categorization of all the different types of *spaces*. This could be anything from classifying and categorizing all different groups to manifolds to algebras. We see the same thing in physics, with talk of symmetries or dualities. If we drop a ball at position A is it the same as dropping a ball at position B? What if we wait five minutes then drop the ball again at position A, will the ball fall the same way? These questions may feel trivial but underneath there is a more fundamental question, which of course extends to more complicated problems. Either math or physics, it does not matter, *what is invariant?* is always the question asked.

The *spaces* that will be relevant for us are topological spaces[1]. So, let us ask the question, what are the invariants of topological spaces? As we will soon see, two topological spaces are the same if one can be continuously transformed into the other and back again (continuously). So, we might think of objects in a less rigid way. Perhaps they are made of playdough. In this manner topological spaces can be classified by how many holes it has: its genus. A sphere has no holes where a torus has one hole, so there is no way to continuously (without cutting, tearing, etc.) transform a sphere into a torus or vice versa. Of course there is also the famous story of the coffee cup and the doughnut, which are both genus 1 so they are topologically equivalent.

[1]This is actually quite broad, as we will see, so, more specifically differentiable manifolds, but we will talk of topological spaces for now.

The story of genus is probably quite familiar. There is, however, a different way to classify spaces that may be a less familiar story. It is done by finding the intersections of two closed curves over the space. Let us go back to the example of the sphere and the torus. Consider two closed curves on the sphere (say two rubber bands on a ball), these curves can be continuously altered such that they never intersect (just pull the rubber bands away from each other). So the sphere's intersection number is trivial. On the torus this is not the case. We have the following situation illustrated below.



As we can see from the above picture there is no way to continuously deform the cyan or red path such that they no longer intersect. So, the intersection number on the torus is one (the paths intersect once).

So, how exactly can we study math through physics? This is done by formulating a topological quantum field theory. These theories are incredibly powerful and provide a bridge between the worlds of math and physics. When we say topological we mean that the theory is somehow global in nature. That is it does not depend on local structure[2] (Think back to the above example about spaces being made of playdough). So, given a space in which a QFT can be defined, the space can be warped and changed (in a continuous manner) without affecting the results of the theory. This should be reflected in the outputs of the theory (the correlation functions), which it is. Because the theory was topological to begin with the correlators will calculate topological invariants. As we will discover, the appropriate topological quantum field theory will compute intersection numbers on the target space. Thus, spaces can be classified and categorized (provided they support a TQFT) as mathematicians do, but by calculating physical quantities (the correlators).

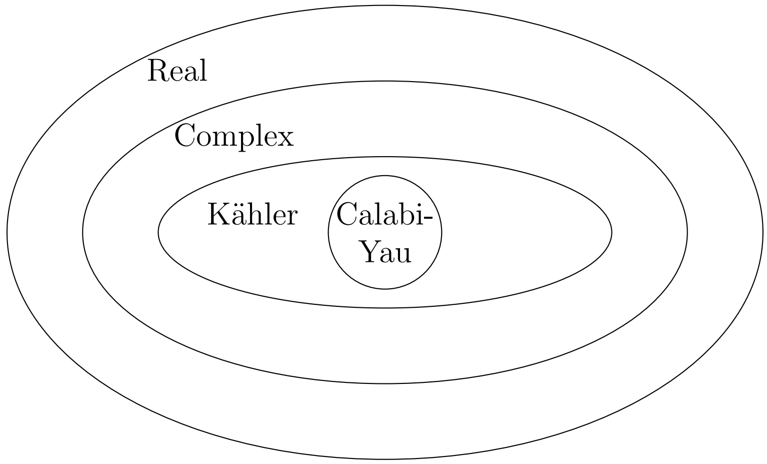
1.2 What Lies Ahead

In this thesis, we journey to a discussion of the topological σ-model and mirror symmetry. We cannot begin our discussion there, however. A mathematical foundation must first be laid. We will do this in two parts: differential geometry and topological quantum field theory. Our discussion of differential geometry will be one guided by our destination of Calabi-Yau manifolds. This means working from real geometry to complex, then to Kähler, and finally to Calabi-Yau. Calabi-Yau manifolds are actually a class of manifolds and they fit into the hierarchy as illustrated in the graphic below[3] (which more or less provides

[2]Of course we will make this much more precise in the coming discussion but it serves as a conceptual baseline for now.

[3]This is to say that Calabi-Yau's are a subset of Kähler manifolds which in turn is a subset of complex manifolds, in turn a subset of real manifolds.

our road map for Chapter 2).



From Calabi-Yau manifolds, in Chapter 3, we will shift our focus to discussing topological quantum field theories and the rich mathematical structure behind them. The key here is connecting the mathematics and physics; to transition from manifolds to field theory. After seeing the inner cogs of TQFTs, we will work to construct one of our own which will comprise the rest of this work. In Chapter 4, we give a refresher on supersymmetry, specifically $\mathcal{N} = (2, 2)$ supersymmetry, which gives rise to the non-linear σ-model (nlσm). So, we work to show the nlσm is a TQFT in Chapter 5 and its relation to mirror symmetry. Finally, we end with a few closing remarks about the constructed TQFT and provide some direction in which this work could be taken. Without further ado, let us dive in head first into differential geometry.

Chapter 2

Differential Geometry and Calabi-Yau Manifolds

Our aim for this chapter is to have an understand Calabi-Yau (CY) manifolds and set up mirror symmetry. Why CY manifolds, why are they important? Considering superstrings in M-theory, we find that the universe must be ten dimensional. Of course we only observe four of these so what do we do with the extra six, where did they go? We go about this problem in a very interesting way. By decomposing the ten dimensional space into a product of a four dimensional space and a six dimensional space, the six dimensional space can be made tiny, curled up, or compactified. Then, supersymmetry constraints imposed by M-theory force the compactified space to be of a very special type. It must be Calabi-Yau.

To begin our study of CY manifolds we will refresh some ideas from differential geometry and expand upon them starting with constructing a differentiable manifold. We will do this through the language of topology and bundles. The construction of manifolds and bundles from topology follows the lectures by F. Schuller [1], and the presentation of complex geometry follows B. Greene's TASI lectures [2], and Nakahara's book [3] quite closley. The section on Calabi-Yau manifolds follows [4, 5]. Some additional points come from [6–8].

2.1 Building Manifolds from a Metric and Bundles

The ideas of topology and metrics gives us an idea of what it means to be close, connected, compact, and continuous which will provide insight into the spaces we want to study. We understand manifolds as topological spaces[1] that are somehow like $\mathbb{R}^n$, which will make this precise momentarily. We can build up to a manifold starting from a very general idea, that is: a metric space. Given a metric space we can induce a topology on that space and with a little more structure we are on our way to a manifold.

[1]The reader is assumed to be familiar with what a topological space is, and can check appendix A for a refresher.

This induced topology is called the *metric topology* and allows us to construct geometries relevant for physics.

Definition 2.1. A set M is a metric space if there is an operation with the set, $d : M \times M \to \mathbb{R}$ where $(x, y) \mapsto d(x, y)$ (usually called a distance function or measure) such that $\forall x, y, z \in M$:

MS0. the function is necessarily nonzero $d(x, y) \geq 0$,

MS1. $d(x, y) = 0 \iff x = y$,

MS2. it is symmetric $d(x, y) = d(y, x)$, and

MS3. the operation satisfies a triangle inequality $d(x, y) \leq d(x, z) + d(z, y)$.

Then (M, d) is a metric space with metric d.

In defining a metric space we now have a notion of distance on a set. Let us get a better feel for this and how this induces a topology through the following example.

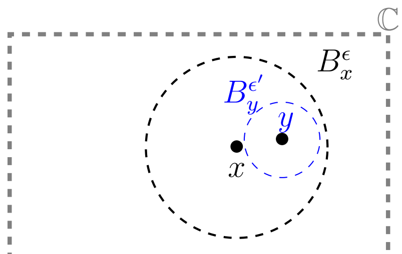
Example 2.1. Let $x, y, z \in X = \mathbb{C}$ (we write $z = a + ib$, $a, b \in \mathbb{R}$) with metric $d : X \times X \to \mathbb{R}$ via $d(x, y) = |x - y|$ where $|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$. We note that the square of a real number is never negative and then confirm this is a metric space:

MS1) $d(x, x) = |(a + ib) - (a + ib)| = 0$, Now if the measure is zero: $0 = |x - y| \implies (x - y)\overline{(x - y)} = 0$, then $x - y = 0 \implies x = y$,

MS2) $d(x, y) = |x - y| = |-1| \cdot |y - x| = d(y, x)$, and we have

MS3) $d(x, y) = |x - y| = |x + z - z - y| \leq |x - z| + |z - y|$.

Now that we have $(\mathbb{C}, d)$ is a metric space let us see what this means about a topology on $\mathbb{C}$. The topology is induced by the metric through neighborhoods called 'epsilon balls': $B_p^\epsilon = \{q \in \mathbb{C} : \text{for } \epsilon > 0 \ |q - p| < \epsilon\}$, then the metric topology is $B_x^\epsilon \in \mathcal{T} \iff \forall y \in B_x^\epsilon \ \exists \ \epsilon' > 0$ s.t. $B_y^{\epsilon'} \subseteq B_x^\epsilon$. It helps to see this with a picture.



We've drawn just one $y \in B_x^\epsilon$ to make the drawing clear but we must draw these epsilon balls for every other point in B_x^ϵ. It is important to note we are free to change ϵ' to make it fit inside B_x^ϵ. If an epsilon ball around $x \in \mathbb{C}$ satisfies this picture *for all* points in the ball then it is allowed in the topology $\mathcal{T}$ of $\mathbb{C}$.

Let's check it is a topological space:

T1) The emptyset trivially enters the topology as it has no elements and is a subset of itself. If we fix an $\epsilon > 0$ then the union of all balls satisfying the requirement to be in the topology will necessarily be all of $\mathbb{C}$ and so $\mathbb{C} \in \mathcal{T}$.

T2) If we take $n < \infty$ balls and look at their intersection, $I = B_1^\epsilon \cap \cdots \cap B_n^\epsilon$, each B_i^ϵ contains an $B_i^{\epsilon'}$ let $B_{min} = \min_{1 \le i \le n}(B_i^{\epsilon'})$ then $B_{min} \subset I$ and hence $I \in \mathcal{T}$.

T3) Let $U = \bigcup_{i \in \mathbb{C}} B_i^\epsilon$, take B_1^ϵ it has $B^{\epsilon'}$ inside it as $B_1^\epsilon \in \mathcal{T} \implies B^{\epsilon'} \subset U$ and hence $U \in \mathcal{T}$.

Thus $(\mathbb{C}, \mathcal{T})$ is a topological space with the metric topology induced by its metric d.

So we understand now that given a metric space (M, d) we not only have to metric structure but also the metric topology, $\mathcal{T}_d$, and so $(M, d, \mathcal{T}_d)$ becomes a topological space as well. We note that every metric space is a Hausdorff[2] topological space.

Example 2.2. Take a look back at Example 2.1, this topological space is Hausdorff as we are free to pick epsilon such that $B_x \cap B_y = \emptyset$.

We're nearly ready to define what a manifold is but need a little more structure on our Hausdorff topological space. As we are familiar with compactness[3], we know a manifold does not need to be compact but it must satisfy a looser condition that of paracompactness. To understand paracompactness we need to *refine* the covering of the space in a special way.

Definition 2.2. Let $(X, \mathcal{T})$ be a topological space with cover, C. A refinement of C is a cover V s.t. $\forall\, W \in V\ \exists\, D \in C$ s.t. $W \subseteq D$. V is called a refinement of C. The refinement V is open if $V \in \mathcal{T}$ and is locally finite if $\forall\, p \in X\ \exists$ a neighborhood, U_p, of p s.t. $\{U \cap U_p | U \in V\}$ has a finite number of elements (finite set).

Definition 2.3. A topological space, $(A, \mathcal{O})$, is paracompact if every cover has a locally finite refinement.

Note that every compact space is paracompact as a sub-cover is a refinement of the original cover.

With these ideas in place we are now ready to shift our focus to manifolds, we start with more definitions.

[2]See Definition A.2.
[3]See Definition A.5.

Definition 2.4. A paracompact Hausdorff topological space $(X, \mathcal{T})$ is a d-dimensional topological manifold if $\forall\ p \in X\ \exists$ a neighborhood of p, $U(p) \in \mathcal{T}$ and a homeomorphism $h : U(p) \to \mathbb{R}^d$.

This is close to the definition we are familiar with but not quite the same. We do not have a differentiable structure yet. Before adding this structure we give an account of bundles.

Definition 2.5. A bundle of a topological manifold is the collection (E, π, M). Where E is the total space (a topological manifold), M is the base space (a topological manifold), π is a continuous surjection, $\pi : E \to M$ called the projection. Let $p \in M$ then the fiber, F at p is $\text{preim}_\pi(\{p\})$[4]. A bundle may also be denoted as $E \xrightarrow{\pi} M$.

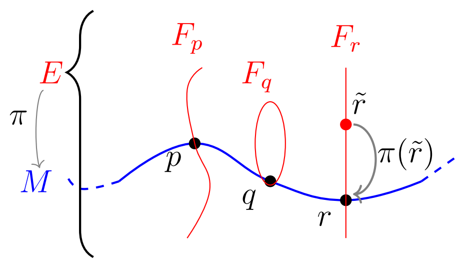


Figure 2.1: A bundle with base space M, total space the collection of all the fibers E, fibers $(F_p,\ F_q,\ F_r)$ over each point in M, and projection map $\pi : E \to M$. Note that the fibers here have arbitrary representation as a line or curve but are not necessarily a line; they can be any object such as a point, a line, a manifold, a vector space, etc.

Definition 2.6. Let (E, π, M) a bundle and F a topological manifold. If $\forall p \in M$ $\text{preim}_\pi(\{p\}) \cong F$ then $E \xrightarrow{\pi} M$ is a fiber-bundle with 'typical fiber' F. If the fiber-bundle of a topological manifold is one dimensional it may be referred to as a line-bundle.

Example 2.3. The $\mathbb{C}$-line bundle over M is a fiber bundle $E \xrightarrow{\pi} M$ with typical fiber $\mathbb{C}$.

Definition 2.7. A section (of a bundle (E, π, M)) is a map $\sigma : M \to E$ s.t. $\pi \circ \sigma = \text{id}_M$ where id_M is the identity map on M.

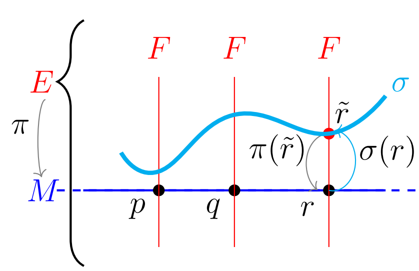
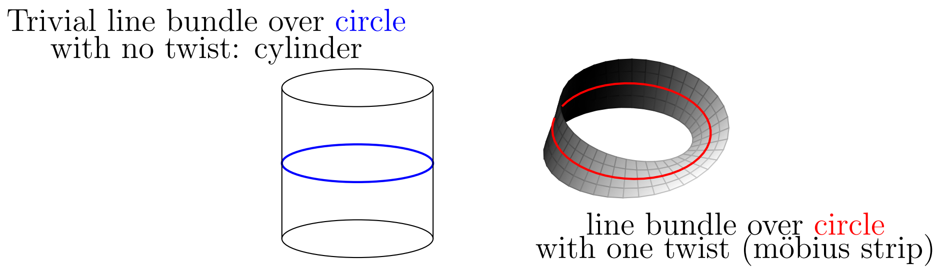


Figure 2.2: A fiber bundle, $E \xrightarrow{\pi} M$, with section, σ. Again the lines are just a representation however this time it is important that they are all 'lines'.

[4]Denoted F_p.

Definition 2.8. We say a fiber-bundle, with fiber F and base space M, is trivial if the fiber-bundle over M (that is the total space E) is isomorphic to the product bundle: $E \cong M \times F$.

Example 2.4. Let M and F be topological manifolds with $m \in M$, $f \in F$. We can consider then the product manifold as the total space $E = M \times F$, and therefore a trivial bundle, with a projection $\pi : M \times F \to M$ where it maps $(m, f) \mapsto m$. If we took $M = S^1$ and $F = \mathbb{R}$ the cylinder would be a trivial bundle as $E = S^1 \times \mathbb{R}$. As another example we could take the Möbius strip as the total space E with $M = S^1$ the circle where $\text{preim}_\pi(\{p\}) = [-1, 1]$ is a bundle. The Möbius bundle is not a trivial bundle.



Example 2.5. In the special case where we have a product (trivial) bundle: $E = M \times F$ (M, F topological manifolds with projection π_1) and an arbitrary map $s : M \to F$, we can take a section $\sigma : M \to M \times F$ s.t. $p \mapsto (p, s(p))$. Take $M = \mathbb{R}^3$ over which a Hilbert space is defined, then referring back to example 2.3, in which the $\mathbb{C}$-line bundle is trivial, the wavefunction is just a section of the trivial $\mathbb{C}$-line bundle, $\mathbb{R}^3 \times \mathbb{C} \xrightarrow{\pi} \mathbb{R}^3$ [1, 3].

Sections are really only defined locally, that is over some open set $U \in \mathcal{T}$ of M. If it is possible to extend the local section over the whole manifold then we obtain a global section. This can be done if the fiber-bundle is trivial, which makes trivial bundles nice to work with.

2.2 Differentiable Manifolds and Bundles

Now in order to get to Calabi-Yau manifolds we need to have some differentiable structure, that is we need to be able to talk about derivatives on the manifold. This leads us into the subject of differential geometry. With the foundation of a topological manifold we can now specify what is meant by differentiable manifold.

Definition 2.9. $(M, \mathcal{T})$ is a d-dimensional differentiable manifold if:

M1. $(M, \mathcal{T})$ is a topological manifold,

M2. the set $\{U(p) \in \mathcal{T}\}$ covers M, that is $M = \bigcup_{p \in M} U(P)$, and

M3. if $U(p), U(q) \in \mathcal{T}$ such that $U(p) \cap U(q) \neq \emptyset$, the corresponding homeomorphisms, h_p, h_q can be composed forming a map $\psi_{pq} = h_p \circ h_q^{-1} : h_q\big(U(p) \cap U(q)\big) \to h_p\big(U(p) \cap U(q)\big)$ that is infinitely differentiable (*smooth* or C^∞).

We call the ψ_{pq} transition functions. If these are satisfied then we say M is a real manifold.

With smooth manifolds in hand we are finally ready to talk about a very special type of bundle.

Definition 2.10. A vector-bundle is a fiber-bundle where the typical fiber is a vector space. This naturally extends to tensors as well.

Vector bundles are incredibly fundamental objects. We wont go into detail here but one can discuss principal bundles and then their associated bundles grossly important in gauge theories and they show up everywhere [1, 9]. Note that every vector space has a dual and so for any vector-bundle we consider over a manifold, we also have the dual vector bundle we can illustrate this through these examples of vector bundles.

Example 2.6. The tangent-bundle of a manifold is the disjoint union of all the tangent spaces:

$$TM = \coprod_{p \in M} T_pM$$
$$\pi : TM \to M$$
$$X \mapsto p$$

where $X \in T_pM$ (the fiber of p in total space TM). Since the total space $E = TM$ is a vector space we also have the dual vector space $E^* = T^*M$ over a manifold M. We can then construct the cotangent-bundle via $E = T^*M$, M, and $\pi : T^*_M \to M$ where a $w \in T^*_pM$, $\pi : w \mapsto p$. Here T^*M is the set of linear maps from $TM \to \mathbb{R}$, so $T^*_pM \equiv \mathrm{Hom}(T_pM, \mathbb{R})$.

2.2.1 Metric, Connection, and Curvature

We are familiar with the ideas of a *Riemann metric, covariant derivative, connection* and *curvature* but we can rephrase them in terms of bundles and sections. As we mentioned before a metric space induces a topological space, well it turns out that the metric, d of the metric space, induces the Riemann metric, g of the manifold [10]. For brevity, we give the definitions of these ideas phrased in terms of bundles.

Definition 2.11. Let $(T^*M \otimes_s T^*M, \pi, M)$ be the symmetric product cotangent bundle. A metric, g, on M is a *global section* of $T^*M \otimes_s T^*M$. Written in coordinates this is

$$g = g_{ab} dx^a \otimes dx^b. \tag{2.1}$$

Writing the metric in this way makes clear $g_{ab} = g_{ba}$, and the metric is said to be defined on the bundle. A connection gives a notion of what it means to be parallel transported on a fiber bundle. This allows us to identify or connect fibers. It is specified through a covariant derivative.

Definition 2.12. Let (E, π, M) be a bundle with total space, E, base space, M, a differentiable manifold, and projection π. The space of smooth sections of the bundle is denoted $\Gamma(E)$. A covariant derivative is a smooth linear map

$$\begin{aligned} &\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E) \quad \text{s.t.} \\ &\nabla(fs) = df \otimes s + f\nabla s, \end{aligned} \tag{2.2}$$

where f is a smooth function on M (lives in T^*M)[5] and s is a smooth section on the total space E.

If total space, E, is a collection of vector spaces over M and therefore a vector (fiber) bundle (the tangent bundle for example), then a smooth section on E would be a vector field over the manifold, we write $X^\mu \frac{\partial}{\partial x^\mu} \in \Gamma(TM)$. Likewise, a smooth function on M is also a section, $f \in \Gamma(M)$, a covector field (forms) over a smooth manifold also gives a smooth section, a one-form $w \in \Gamma(T^*M)$ (Of the cotangent bundle). Take p copies of the cotangent bundle to obtain the space of p-forms, $\bigwedge^p T^*M$, which is also a bundle over M. If β is a p-form, $\beta \in \Gamma\big(\bigwedge^p T^*M\big)$.

The covariant derivative differentiates sections of a bundle along tangent directions to the base manifold, M. The connection is then specified by parallel sections being trivial under the covariant derivative. It is said to be a connection on a bundle.

Definition 2.13. Let $\nabla : v^a \mapsto \partial_i v^a + (A_i)^a{}_b v^b$, then the connection is $(A_i)^a{}_b$. It is a linear map[6] valued one form.

We can write the covariant derivative in nice shorthand as $D = d + A$. Now we know holonomy as the study of tangent vectors being parallel transported around a closed curve on a manifold. When the vector comes back rotated with was fundamentally do to some curvature of the manifold and thus we get curvature from holonomy. That is curvature is the measure of parallel transporting a vector around a closed loop on M.

Definition 2.14. Given a tangent bundle over a smooth manifold the curvature of the manifold is a linear map

$$\begin{aligned} &R : \Gamma(TM) \to \Gamma(TM) \\ &R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \end{aligned} \tag{2.3}$$

We can think of this map as putting in two vectors and then outputting an infinitesimal rotation. We can shorthand this like the curvature as $R = dA + A \wedge A$.

[5] A 0-form.

[6] Usually called an endomorphism valued one form.

Given a bundle that we can define curvature map on with a connection defined over the total space, we can probe the topology of the bundle through the Chern classes.

Definition 2.15. Let (E, π, M) be a differential complex vector bundle, M, differentiable manifold, with curvature two-form $R = \mathrm{d}A + A \wedge A$, the curvature of connection A on total space E. The total Chern class of E, $c(E)$ is

$$\begin{aligned} c(E) &= det\Big(1 + \frac{i}{2\pi}R\Big) \\ &= 1 + \frac{i}{2\pi}Tr(R) - \frac{1}{4\pi^2}Tr(F \wedge F - 2(Tr(R))^2) + \dots \\ &= c_0(E) + c_1(E) + c_2(E) + \dots \in H^0 \oplus H^2 \oplus H^4 \oplus \dots \end{aligned}$$

where $c_{i>k}(E) = 0$ and $c_i(E) \in H^{2i}(M)$ k is the dimension of the total space and H^{2i} is the $2i^{th}$ De Rahm cohomology group. The Chern class of the tangent bundle is also referred to as the Chern class of the manifold itself.

The top Chern class is the cycle associated to a generic section so the top Chern class represents the intersection of a generic section with the zero section [9]. Chern classes are also a measure of how close the bundle is to being trivial or not [4].

We will not need the following until much later (Chapter 5), but here is a good a place as any for introductions. One may wish to rewrite the Chern class as a series. This can be done through the Chern character.

Definition 2.16. Let (E, π, M) be as in definition 2.15, with the dimension of $E = n$. If $\exists\ x_i$ s.t. $c(E) = \prod_{i=1}^{n}(1 + x_i)$ then the Chern Character has just been defined as $ch(E) = \sum_i e^{x_i}$. This can be expanded as

$$ch(E) = n + c_1(E) + \frac{1}{2}(c_1^2(E) - 2c_2(E)) + \cdots . \tag{2.4}$$

This naturally leads the Todd Class.

Definition 2.17. Again taking the same assumptions from 2.15 and 2.16, the Todd Class of the total space, E, is given by,

$$td(E) = \prod_{i=1}^{n} \frac{x_i}{1 - e^{-x_i}} = 1 + \frac{1}{2}c_1(E) + \frac{1}{12}(c_1^2(E) + c_2(E)) + \cdots . \tag{2.5}$$

Now, given both the Chern character and the Todd Class of a bundle we can actually find the bundles' Euler character due to Grothendieck, Riemann, and Roch. One of the many Euler character formulas is given by [9], for some bundle (E, π, M) and defined cohomology group $H^k(E)$, $\chi(E) = \Sigma_k(-1)^k \mathrm{dim}(H^k(E))$. The Grothendieck-Riemann-Roch formula computes then,

$$\chi(E) = \sum_k (-1)^k \mathrm{dim}(H^k(E)) = \int_M ch(E) \wedge td(M). \tag{2.6}$$

Equation 2.6 does not have much relevance to us right now, but will come Chapter 5. We will see this appear when we are ready to discuss the results of topological quantum field theory.

Having taken a closer look at differential geometry we are now ready to look at complex geometry. Strictly real differentiable manifolds have many nice properties, but not nearly enough to be considered for a compactified theory. We need to refine our search further, starting with complex manifolds.

2.3 Complex Geometry: Kähler and Calabi-Yau

Real manifolds are nice but the work we will do is based upon complex geometry as Kähler and Calabi-Yau manifolds are complex.

Definition 2.18. M is an n-dimensional complex manifold if M is a differentiable manifold, the homeomorphisms $h_p : U(p) \to \mathbb{C}^n$, and the transition functions are holomorphic[7].

Example 2.7. Some examples of complex manifolds are S^2 and T^2. Here we illustrate the complex nature of S^2 through the stereographic projection.

To see that S^2 is a complex manifold, start with the familiar stereographic projection of the real manifold S^2 embedding ($\hookrightarrow$) in $\mathbb{R}^3$ mapping to $\mathbb{R}^2$, such that the real plane bisects the 2-sphere at the equator and the north pole located at $N = \langle 0, 0, 1\rangle$ and south pole located at $S = \langle 0, 0, -1\rangle$. We define the north pole chart as the (invertible) map[a] $\pi_N : \left(\frac{x}{1-z}, \frac{y}{1-z}\right) \mapsto (a, b)$, where $a, b \in \mathbb{R}^2$, $x, y, z \in S^2 \hookrightarrow \mathbb{R}^3$. Likewise the south pole chart is $\pi_S : \left(\frac{x}{1+z}, \frac{-y}{1+z}\right) \mapsto (c, d)$ where $c, d \in \mathbb{R}^2$, and we know in this real case the transition functions are differentiable.

Now make the manifold complex by complexifying $\mathbb{R}^2$ in the usual way. On the north pole chart we have $u = a + ib$, $\bar{u} = a - ib$, $v = c + id$, and $\bar{v} = c - id$, $u, \bar{u}, v, \bar{v} \in \mathbb{C}$. Then, since a, b, c, d defined coordinates on S^2 so must $u, \bar{u}, v$, and $\bar{v}$. Now to check the transition function are holomorphic we write try to find $v = v(u, \bar{u})$. Begin by noticing

$$\frac{u}{\bar{u}} = \frac{x+iy}{x-iy} = \frac{\bar{v}}{v}$$
$$\implies v = \frac{\bar{u}}{u}\bar{v} = \frac{x-iy}{x+iy}\frac{x+iy}{1+z} = \frac{x^2+y^2}{(x+iy)(1+z)}.$$

Now since u, v and there conjugates live on the sphere we know their modulus must be unitary,

[7] See Definition A.6.

$|u| = |v| = 1$, as well as $\frac{|u|}{|v|} = 1$.

$$\begin{aligned}1 = |u| &= \frac{x+iy}{1-z}\frac{x-iy}{1-z} \implies (1-z)^2 = x^2+y^2 \\ = |v| &= \frac{x+iy}{1+z}\frac{x-iy}{1+z} \implies (1+z)^2 = x^2+y^2 \\ \implies & \frac{(1-z)^2}{(1+z)^2} = 1.\end{aligned}$$

Therefore, returning to building the transition function,

$$v = \frac{x^2+y^2}{(x+iy)(1+z)} = \frac{(1-z)^2}{(x+iy)(1+z)} = \frac{1}{u}.$$

Thus the transition function $v = v(u, \bar{u})$ is holomorphic, as $\frac{\partial v}{\partial \bar{u}} = 0$, and S^2 is a complex manifold.

[a] For a reminder of how this is derived consult example A.3.

Definition 2.18 presupposes a lot of structure to make a manifold complex. A motivating question then is when does a real manifold become a complex manifold. We can actually build a complex manifold from a real one much the same way we build complex numbers from real numbers. This construction follows that of [3].

2.3.1 Complex Tangent Bundle and Almost $\mathbb{C}$-Structure

Just as complexification of $\mathbb{R}^2$ into $\mathbb{C}$ is done by taking $x, y \in \mathbb{R}$ and making $z = x+iy \in \mathbb{C}$, we can complexify the tangent bundle, $TM \to TM^{\mathbb{C}}$, in a similar way. That is if we take $\vec{V}, \vec{W} \in T_pM$ we squish them together to obtain $\vec{Z} = \vec{V} + i\vec{W} \in T_pM^{\mathbb{C}} = T_pM \otimes \mathbb{C}$. Then, just as for TM, $TM^{\mathbb{C}} = \coprod_{p\in M} T_pM^{\mathbb{C}}$. At this stage it is hard to see much of a reason for this but by installing a *complex structure* on M the benefits become clear.

Definition 2.19. Let M be a d-dimensional differentiable manifold with d even. The almost $\mathbb{C}$-structure is a map $J : TM^{\mathbb{C}} \to TM^{\mathbb{C}}$ s.t. $J^2 = -I_{T_pM}$. We say (M, J) is an almost complex manifold.

This almost complex structure does something wonderful, it decomposes the tangent bundle into holomorphic and antiholomorphic tangent spaces. This decomposition becomes evident through inspection of the eigenvalues of J; they are $\pm i$. The complexified tangent space decomposes into two disjoint subspaces corresponding to the eigenvalues: $T_pM^{\mathbb{C}} = T_pM \otimes \mathbb{C} = \underbrace{T^{1,0}}_{+i} \oplus \underbrace{T^{0,1}}_{-i}$. We say vector fields $Z \in T^{1,0}$ (those with eigenvalue $+i$) are holomorphic and those $\bar{Z} \in T^{0,1}$ (with eigenvalue $-i$) are anti-holomorphic[8]. It is

[8] We've shorthand denoted $T^{1,0}$ $(T^{0,1}) = TM^{1,0}$ $(TM^{0,1})$, and holomorphic vectors live in $T_p^{1,0}$ (anti-holomorphic in $T_p^{0,1}$).

natural then to define a projection operator to project onto either subspace of $T_pM^{\mathbb{C}}$:

$$\begin{aligned}
&\mathcal{P}^+ : TM^{\mathbb{C}} \to T^{1,0} \\
&\mathcal{P}^- : TM^{\mathbb{C}} \to T^{0,1} \\
&\mathcal{P}^{\pm} = \frac{1}{2}(I \mp iJ).
\end{aligned} \tag{2.7}$$

We define this in such a way that $\forall\ Z \in TM^{\mathbb{C}},\ \mathcal{P}^+Z \in T^{1,0},\ \mathcal{P}^-Z \in T^{0,1},\ \mathcal{P}^{\pm}\mathcal{P}^{\mp} = 0$ as one can check. We want our manifolds to be complex not just *almost* complex. In order to get to a complex manifold we need the almost $\mathbb{C}$-structure to be integrable.

Definition 2.20. If (M, J) is an almost complex manifold s.t. the lie bracket of *any* two holomorphic vector fields is again a holomorphic vector field, $[T^{1,0}, T^{1,0}] \subset T^{1,0}$, then J is integrable.

That is to say the complex structure is preserved along the manifold. This leads us to a complex manifold through the Nijenhuis tensor.

Definition 2.21. Let (M, J) be an almost complex manifold. The Nijenhuis tensor is a map

$$\begin{aligned}
&N : TM \times TM \to TM \\
&N(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY]
\end{aligned} \tag{2.8}$$

If the Nijenhuis tensor vanishes then the almost complex structure is integrable. It also means that the almost complex manifold becomes a complex manifold. This is due to a theorem by Newlander and Nirenburg.

Theorem 2.1 (Newlander-Nirenburg)**.** *Let (M, J) be an almost complex manifold. M is a complex manifold if and only if the J is integrable.*

Proposition 2.1. If the almost complex structure is constant it is integrable and the Nijenhuis tensor vanishes.

Solution. Let $X, Y \in T^{1,0}$, then we can write $X = \mathcal{P}^+Z,\ Y = \mathcal{P}^+\tilde{Z}\ \forall\ Z, \tilde{Z} \in TM$. Then

$$\begin{aligned}
[X, Y] &= [\mathcal{P}^+Z, \mathcal{P}^+\tilde{Z}] \\
&= \mathcal{P}^+\mathcal{P}^+[Z, \tilde{Z}] + \mathcal{P}^+[Z, \mathcal{P}^+]\tilde{Z} + [\mathcal{P}^+, \mathcal{P}^+]Z\tilde{Z} + \mathcal{P}^+[\mathcal{P}^+, \tilde{Z}]Z \\
&= \mathcal{P}^+\Big([Z, \tilde{Z}] + [Z, \mathcal{P}^+]\tilde{Z} + [\mathcal{P}^+, \tilde{Z}]Z\Big) \in T^{1,0}.
\end{aligned}$$

Thus constant J is integrable, and a similar calculation shows

$$\begin{aligned}
N(X, Y) &= [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY] \\
&= [X, Y] - [X, Y] + J[J, Y]X - [X, Y] + J[X, J]Y + [X, Y] - J[J, Y]X - J[X, J]Y \\
&= 0,
\end{aligned}$$

the Nijenhuis tensor vanishes. $\square$

Thus, we can build a complex manifold from a real one with an appropriate integrable almost complex structure. On a complex manifold the coordinates $\{z_1, \ldots, z_n\} \in \mathbb{C}$ and the holomorphic tangent space is given by $T^{1,0} = \mathrm{Span}_{\mathbb{C}}\{\frac{\partial}{\partial z^1}, \cdots, \frac{\partial}{\partial z^n}\}$, and the antiholomorphic tangent space is $T^{0,1} = \mathrm{Span}_{\mathbb{C}}\{\frac{\partial}{\partial \bar{z}^1}, \cdots, \frac{\partial}{\partial \bar{z}^n}\}$. In summary, a $2d$-dimensional real differentiable manifold with integrable almost complex structure is a complex manifold.

2.3.2 Differential Forms: From De Rahm to Dolbeault Cohomology

If J is integrable, making M complex, complex coordinates can be prescribed to the manifold. Then just as vector fields over the complex manifold decomposed into holomorphic and anti-holomorphic vectors, we should expect forms over the manifold to decompose into holomorphic and anti-holomorphic as well. We denote $\Omega^p(M)$ as the space of p-forms fields on[9] M (also denoted $\bigwedge^p T^*$). The space of (p,q)-form fields over M is denoted $\Omega^{p,q}(M)$ where elements have p holomorphic indices and q anti-holomorphic indices. The decomposition is then:

$$\Omega^k(M) = \bigoplus_{p+q=k} \Omega^{p,q}(M).$$

From this the space of (p,q)-forms is the conjugation of the space of (q,p)-forms $\Omega^{p,q}(M) = \overline{\Omega^{q,p}(M)}$. For example the space of 2-forms at $x \in M$ decomposes as $\Omega^2_x(M) = \Omega^{2,0}_x(M) \oplus \Omega^{1,1}_x(M) \oplus \Omega^{0,2}_x(M)$. A (p,q)-form on M is denoted

$$\alpha = \alpha_{\mu_1 \cdots \mu_p \bar{\nu}_1 \cdots \bar{\nu}_q}(z, \bar{z}) dz^{\mu_1} \wedge \cdots \wedge dz^{\mu_p} \wedge d\bar{z}^{\bar{\nu}_1} \wedge \cdots \wedge d\bar{z}^{\bar{\nu}_q}. \tag{2.9}$$

The exterior derivative for real manifolds decomposes as the space on which it acts decomposes. If $\alpha \in \bigwedge^p T^*$ that is α is a $(p = q + r)$-form,

$$\begin{aligned} \mathrm{d} &: \bigwedge^p T^* \to \bigwedge^{p+1} T^* \\ \mathrm{d} &: \alpha \mapsto \mathrm{d}\alpha, \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} \mathrm{d}\alpha =& \frac{\partial}{\partial z^\rho}\left[\alpha_{\mu_1 \cdots \mu_p \bar{\nu}_1 \cdots \bar{\nu}_q}(z, \bar{z})\right] dz^\rho \wedge dz^{\mu_1} \wedge \cdots \wedge dz^{\mu_p} \wedge d\bar{z}^{\bar{\nu}_1} \wedge \cdots \wedge d\bar{z}^{\bar{\nu}_q} \\ &+ (-1)^p \frac{\partial}{\partial \bar{z}^{\bar{\sigma}}}\left[\alpha_{\mu_1 \cdots \mu_p \bar{\nu}_1 \cdots \bar{\nu}_q}(z, \bar{z})\right] dz^{\mu_1} \wedge \cdots \wedge dz^{\mu_p} \wedge d\bar{z}^{\bar{\sigma}} \wedge d\bar{z}^{\bar{\nu}_1} \wedge \cdots \wedge d\bar{z}^{\bar{\nu}_q}. \end{aligned} \tag{2.11}$$

Equation 2.11 makes clear the decomposition of d as:

$$\begin{aligned} \mathrm{d} &= dz^\mu \wedge \frac{\partial}{\partial z^\mu} + d\bar{z}^{\bar{\nu}} \wedge \frac{\partial}{\partial \bar{z}^{\bar{\nu}}} \\ &= \partial + \bar{\partial} \end{aligned}$$

[9] We assume M is complex now.

where we have just defined $\partial \equiv$ holomorphic exterior derivative and $\bar{\partial} \equiv$ anti-holomorphic exterior derivative, where

$$\begin{aligned} \partial &: \Omega^{p,q} \to \Omega^{p+1,q}, \\ \bar{\partial} &: \Omega^{p,q} \to \Omega^{p,q+1}. \end{aligned} \tag{2.12}$$

The decomposition and maps to higher forms over M is,

$$\begin{tikzcd}
\Omega^0: \arrow[d, "\mathrm{d}"'] & & & \Omega^{0,0} \arrow[dl, bend right, "\partial"'] \arrow[dr, bend left, "\bar{\partial}"] & & \\
\Omega^1: & & \Omega^{1,0} \arrow[dl] \arrow[dr] & \oplus & \Omega^{0,1} \arrow[dl] \arrow[dr] & \\
\Omega^2: & \Omega^{2,0} & \oplus & \Omega^{1,1} & \oplus & \Omega^{0,2}
\end{tikzcd}$$

Just as $\mathrm{d}^2 = 0$, it is natural to ask the same for ∂ and $\bar{\partial}$.

Proposition 2.2. The following operators are nilpotent: ∂, $\bar{\partial}$, and $(\partial\bar{\partial} + \bar{\partial}\partial)$.

Solution. We show that $\partial^2 = \partial\bar{\partial} + \bar{\partial}\partial = \bar{\partial}^2 = 0$ using that $\mathrm{d}^2 = 0$. First note that partial derivatives commute, but we have the anti-symmetric product in forms so $\partial^2 = 0 = \bar{\partial}^2$. Therefore, $0 = \mathrm{d}^2 = (\partial^2 + \partial\bar{\partial} + \bar{\partial}\partial + \bar{\partial}^2) = (\partial\bar{\partial} + \bar{\partial}\partial)$. Thus the operators are nilpotent and we have that $\{\partial, \bar{\partial}\} = 0$. $\square$

Originally the nilpotency of the exterior derivative allowed us to define the De Rahm cohomology as forms which were closed but not exact. However, now that it decomposes, two new differential operators exist and proposition 2.2 means and they each define new cohomology groups called the *Dolbeaut cohomology*.

Definition 2.22. Let $\alpha \in \Omega^{p,q}(M)$, the Dolbeaut cohomology groups are defined as

$$\begin{aligned} H^{p,q}_{\partial}(M) &= \frac{\{\alpha \mid \partial\alpha = 0\}}{\{\alpha = \partial\beta \mid \beta \in \Omega^{p-1,q}\}}, \\ H^{p,q}_{\bar{\partial}}(M) &= \frac{\{\alpha \mid \bar{\partial}\alpha = 0\}}{\{\alpha = \bar{\partial}\beta \mid \beta \in \Omega^{p,q-1}\}}. \end{aligned} \tag{2.13}$$

We will use the anti-holomorphic Dolbeaut cohomology group. The $\dim(H^{p,q}(M)) = h^{p,q}$, and they are named the Hodge numbers.

Recall that on real manifolds with a metric, the exterior derivative has an adjoint, $d^\dagger$, defined as (on a Lorentzian manifold)

$$\begin{aligned} \mathrm{d}^\dagger &: \Omega^r \to \Omega^{r-1}, \\ \mathrm{d}^\dagger &= (-1)^{mr+m} * \mathrm{d} * . \end{aligned} \tag{2.14}$$

Where we use the Hodge star operator on an r-form, β, as

$$*\beta = \frac{\sqrt{|g|}}{r!(m-r)!}\beta_{\mu_1\cdots\mu_r}g^{\mu_1\nu_1}\cdots g^{\mu_r\nu_r}\epsilon_{\nu_1\cdots\nu_m}dx^{\nu_{r+1}}\wedge\cdots\wedge dx^{\nu_m}.$$

Which leads to a Laplacian operator, $\Delta = \{\mathrm{d}, \mathrm{d}^\dagger\}$. Furthermore, we remember, on a compact Riemannian manifold any form can be decomposed uniquely, via the Hodge decomposition, to the sum of *exact*, *coexact*, and *harmonic* pieces. The key idea being that the harmonic term is a unique representative of the $H^p(M)$ cohomology group. It comes as no surprise by now that on a complex manifold the Hodge decomposition decomposes further as both d and $\mathrm{d}^\dagger$ split. Since we make this discussion in terms of $H^{p,q}_{\bar\partial}(M)$, a (p,q)-form on M

Theorem 2.2 (Hodge)**.** *Let M be a complex manifold with $\alpha \in \Omega^{p,q}(M)$. Then α can be uniquely written as*

$$\alpha^{p,q} = \bar\partial\beta^{p,q-1} + \bar\partial^\dagger\gamma^{p,q+1} + \tilde\alpha^{p,q}. \tag{2.15}$$

Where $\Delta_{\bar\partial}\tilde\alpha^{p,q} = 0$, an antiholomorphic harmonic form.

On a complex manifold the Laplacian is

$$\begin{aligned}\Delta = \{\mathrm{d}, \mathrm{d}^\dagger\} &= \{(\partial + \bar\partial), (\partial^\dagger + \bar\partial^\dagger)\} \\ &= \{\partial, \partial^\dagger\} + \cancelto{0}{\{\partial, \bar\partial^\dagger\}} + \cancelto{0}{\{\bar\partial, \partial^\dagger\}} + \{\bar\partial, \bar\partial^\dagger\} \\ &= \Delta_\partial + \Delta_{\bar\partial},\end{aligned} \tag{2.16}$$

as holomorphic and antiholomorphic do not mix. Already the structure of a complex manifold has done a lot of nice work for us. As we will see, if we restrict to a special subset of complex manifolds, those that are Kähler, they become a natural space on which to study elementary physics.

Before moving onto Kähler manifolds let us shift our focus back to bundles for a moment. Now that we have a little more familiarity with complex manifolds we can talk about a special bundle that will be important for later.

Definition 2.23. The holomorphic vector bundle, $E \xrightarrow{\pi} M$, is a fiber-bundle where the base space, M, is a complex manifold, the total space, E, has a complex structure, typical fiber is $\mathbb{C}^k$, the projection, π, is a holomorphic map, and there is a biholomorphic[10] section.

Note the biholomorphic section makes a trivialization of the bundle for open subsets of M.

Example 2.8. The trivial vector bundle $E \cong M \times \mathbb{C}^k$ over M is a holomorphic vector bundle. Since the complexified tangent spaces admits complex structure, $TM^{\mathbb{C}}$, $T^*M^{\mathbb{C}}$, are holomorphic vector bundles. Recall the space of forms, $\Omega^p(M)$ defines a bundle over M, when the space decomposes, only the bundles with no anit-holomorphic indices are holomorphic vector bundles, e.g. $\Omega^{r,0}$ is a holomorphic vector bundle.

---

[10] See definition A.7.

Later on we will want to consider holomorphic top forms, these are sections of the $\Omega^{m,0}(M)$ bundle on m-dimensional complex manifold M. We give this bundle a special name when the fibers bundle becomes a line bundle [7].

Definition 2.24. Let (E, π, M) be a holomorphic line-bundle (typical fiber is $\mathbb{C}^1$ instead of arbitrary k), the canonical bundle is $K_M = \bigwedge^{m,0} T^*$ over M.

2.3.3 Kähler Manifolds

With complex manifolds in hand, we are ready to sift through and find which manifolds will lead to Calabi-Yau manifolds. Enter Kähler manifolds. These are special manifolds in which the metric plays nicely with the complex structure and give rise to a special form.

Definition 2.25. A metric g is a hermitian metric if it preserves the complex structure:

$$g(Jx, Jy) = g(x, y), \ \forall \ x, y \in TM \tag{2.17}$$

If g is hermitain, (M, g, J) is a hermitain manifold.

Proposition 2.3. Any complex manifold (M, g, J) admits a hermitain metric and is therefore also a hermitain manifold.

Solution. Let $g(x, y)$ be a metric on M. Then, define $\tilde{g}(x, y) = \frac{1}{2}(g(x, y) + g(Jx, Jy))$, and consider

$$\tilde{g}(Jx, Jy) = \frac{1}{2}(g(Jx, Jy) + (-1)^2 g(x, y)) = \tilde{g}(x, y). \tag{2.18}$$

Therefore, the complex manifold admits a hermitain metric and is also a hermitain manifold [3]. □

Proposition 2.4. If g is a hermitian metric, the strictly holomorphic and strictly anti-holomorphic pieces of the metric vanish, $g_{ij} = g_{\bar{i}\bar{j}} = 0$.

Solution.

$$\begin{aligned} g(x, y) = g(Jx, Jy) = J^2 g(x, y) = -g(x, y) &\implies g_{ij} = 0 \\ g(\bar{x}, \bar{y}) = g(J\bar{x}, J\bar{y}) = J^2 g(\bar{x}, \bar{y}) = -g(\bar{x}, \bar{y}) &\implies g_{\bar{i}\bar{j}} = 0. \end{aligned}$$

□

Also observe that real g implies $g_{\bar{i}\bar{j}} = \overline{g_{ij}}, \ g_{\bar{i}j} = \overline{g_{i\bar{j}}}$. This means that the metric can be written as

$$g_{\mu\bar{\nu}} dz^\mu \otimes d\bar{z}^{\bar{\nu}} + g_{\bar{\nu}\mu} d\bar{z}^{\bar{\nu}} \otimes dz^\mu. \tag{2.19}$$

What about considering J on only one vector? In doing so a new form is defined, the Kähler form.

Definition 2.26. The Kähler form, ω, of a hermitian manifold is $\omega(x, y) = g(x, Jy)$.

Proposition 2.5. The Kähler form, ω, lives in $\bigwedge^{1,1} T^*$.

Solution. We know that $g \in T^*M \otimes_s T^*M$. If something were to make this antisymmetric it would become a 2-form. In a sense the Kähler form does this, so by showing ω is antisymmetric we have turned g in the symmetric cotangent space to living in the antisymmetric cotangent space, that is $\bigwedge^2$ and as M is complex we know the space of 2-forms decomposes. We can then read off holomorphic and anti-holomorphic indices.

$$\omega(x, y) = g(x, Jy) = g(Jx, J^2 y) = g(Jx, -y) = g(-y, Jx) = -g(y, Jx) = -\omega(y, x),$$

$$\begin{aligned} \implies \omega &= g_{\mu\bar{\nu}} J dz^\mu \otimes d\bar{z}^{\bar{\nu}} + g_{\bar{\nu}\mu} J d\bar{z}^{\bar{\nu}} \otimes dz^\mu \\ &= i g_{\mu\bar{\nu}} dz^\mu \otimes d\bar{z}^{\bar{\nu}} - i g_{\bar{\nu}\mu} d\bar{z}^{\bar{\nu}} \otimes dz^\mu \\ &= i g_{\mu\bar{\nu}} dz^\mu \wedge d\bar{z}^{\bar{\nu}} \in \Omega^{1,1}(M). \end{aligned} \tag{2.20}$$

□

Definition 2.27. A Kähler manifold, (M, g, ω), is a complex manifold, M, with hermitian metric, g, and closed Kähler form, $\mathrm{d}\omega = 0$.

A few observations about Kähler manifolds.

1. On an n-dimensional Kähler manifold a holomorphic vector will be parallel transported around a closed loop and remain holomorphic and without changing its length. This means its holonomy is $U(d)$, $d \leq n$ [3].
2. The Kähler form is not $\bar{\partial}$-exact so $\omega \in H^{1,1}_{\bar{\partial}}(M)$, this is the *Kähler class*.
3. Equation 2.16 becomes $\Delta = 2\Delta_\partial$, as $\Delta_\partial = \Delta_{\bar{\partial}}$ on a Kähler manifold.
4. $\mathrm{d}\omega = 0 \implies (\partial + \bar{\partial})\omega = 0 \implies \partial\omega = 0$, and $\bar{\partial}\omega = 0$.

Observation number 3 means that the cohomology group of (r, s)-forms is related to the cohomology group of (s, r)-forms through conjugation. So for a Kähler manifold $h^{r,s} = h^{s,r}$. It also tells us that the dimension of the p^{th}-De Rahm cohomology group is related to the dimension of the Dolbeaut cohomology group (the Hodge numbers) as, $\dim(H^p(M)) = \sum_{r+s=p} h^{r,s}$. Then, because the space of forms decomposes on a complex manifold and we have the Hodge dual space, we can organize the space of all forms in a diamond (as shown below for a $\dim_{\mathbb{C}}(M) = 3$).

$$\begin{array}{ccccccc} & & & h^{0,0} & & & \\ & & h^{1,0} & & h^{0,1} & & \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ & & h^{3,2} & & h^{2,3} & & \\ & & & h^{3,3} & & & \end{array}$$

At this point the diamond is not much more than a pretty picture. However, there is much to discovered here if Kähler M satisfies enough special requirements.

Observation number 4, enforces nice conditions on the hermitian metric g. We derived differential conditions on g through this as

$$
\begin{aligned}
0 = \mathrm{d}\omega = i\partial_{[\lambda} g_{\mu]\bar{\nu}} dz^{\lambda} \wedge dz^{\mu} \wedge d\bar{z}^{\bar{\nu}} - i\partial_{[\bar{\lambda}} g_{\bar{\nu}]\mu} dz^{\mu} \wedge d\bar{z}^{\bar{\lambda}} \wedge d\bar{z}^{\bar{\nu}} \\
\text{so, } \partial_{[\lambda} g_{\mu]\bar{\nu}} = 0 \text{ and } \partial_{[\bar{\lambda}} g_{\bar{\nu}]\mu} = 0, \\
\implies \partial_{\lambda} g_{\mu\bar{\nu}} = \partial_{\mu} g_{\lambda\bar{\nu}} \quad \partial_{\bar{\lambda}} g_{\mu\bar{\nu}} = \partial_{\bar{\nu}} g_{\mu\bar{\lambda}}.
\end{aligned}
\tag{2.21}
$$

Equation 2.21 means that the metric can be written as any function which satisfies these conditions.

Definition 2.28. The Kähler potential, $\mathcal{K}$, is a function for which $g_{\mu\bar{\nu}} = \partial_{\mu}\partial_{\bar{\nu}}\mathcal{K}$.

This of course means then the Kähler form can be rewritten as $\omega_{\mu\bar{\nu}} = i\partial_{\mu}\partial_{\bar{\nu}}\mathcal{K}$.

Taking the Levi-Civita connection on M with a hermitian metric means that many of the Christoffel symbols vanish as we only have mixed holomorphic and anti holomorphic indices on g. The nonzero components are

$$
\begin{aligned}
\Gamma^{i}_{jk} = \frac{1}{2} g^{i\bar{l}} (\partial_j g_{\bar{l}k} + \partial_k g_{\bar{l}j} - \cancelto{0}{\partial_{\bar{l}} g_{jk}}) = g^{i\bar{l}} \partial_j g_{k\bar{l}} \\
\Gamma^{\bar{i}}_{\bar{j}\bar{k}} = \frac{1}{2} g^{\bar{i}l} (\partial_{\bar{j}} g_{l\bar{k}} + \partial_{\bar{k}} g_{\bar{j}l} - \cancelto{0}{\partial_{l} g_{\bar{j}\bar{k}}}) = g^{\bar{i}l} \partial_j g_{l\bar{k}}.
\end{aligned}
\tag{2.22}
$$

This means that the Riemann tensor for a Kähler manifold as greatly reduces. The only components which survive are $R^{\bar{\rho}}{}_{\bar{\sigma}\mu\bar{\nu}} = \partial_{\mu}\Gamma^{\bar{\rho}}_{\bar{\nu}\bar{\sigma}}$, so

$$
R_{\rho\bar{\sigma}\mu\bar{\nu}} = g_{\rho\bar{\lambda}} R^{\bar{\lambda}}{}_{\bar{\sigma}\mu\bar{\nu}} = g_{\rho\bar{\lambda}} \partial_{\mu} \Gamma^{\bar{\lambda}}_{\bar{\nu}\bar{\sigma}}, \tag{2.23}
$$

and of course we have nonzero components corresponding to conjugation, and anti-symmetric/symmetric indices swapping. The Ricci tensor is then

$$
R_{\mu\bar{\nu}} = R^{\rho}{}_{\rho\mu\bar{\nu}} = -\partial_{\bar{\nu}} \Gamma^{\rho}_{\mu\rho} = -\partial_{\bar{\nu}} (g^{\rho\bar{\lambda}} \partial_{\mu} g_{\rho\bar{\lambda}}) \tag{2.24}
$$

Having a Kähler manifold means there is much structure to work with. It is a natural space for physics because of this structure as we will see. However, when it comes to compactifying our 10 dimensional theory, it is a necessary requirement that some supersymmetries are persevered. Kähler manifolds, although nice, are not quite good enough to preserve the supersymmetry. We need a special subset of Kähler manifolds, those that are Calabi-Yau.

2.3.4 Calabi-Yau Manifolds

We have finally made it to Calabi-Yau manifolds, these spaces are the most natural for string theory as they come out as eligible spaces after compactification. Without further ado, we define Calabi-Yau:

Definition 2.29. A Calabi-Yau manifold is a compact Kähler manifold that has vanishing first Chern class, $c_1(M) = 0$.

Introducing Chern classes into the discussion brings in a bounty of information about CY manifolds, that make them so special. Firstly, we have a a theorem by Yau.

Theorem 2.3 (Yau)**.** *Let M be a Kähler manifold with Kähler form, ω. If M has vanishing first Chern class, $c_1(M) = 0$, then $\exists!$ a Ricci flat metric, g, and hence a unique Kähler class.*

Two immediate corollaries can be made. First, as CY manifolds are a subset of theorem 2.3's assumptions (those that are compact), every CY manifold is Ricci flat. Second, since Ricci flat metrics are (trivial) solutions to Einstein's vacuum equations, Calabi-Yau manifolds are solutions to $G_{\mu\nu} = 0$ [6]. Of course flatness is nice because it is easy to work with, but it can be increasingly difficult to write an explicit Ricci flat metric. Yau's theorem goes a long way in helping us find flat metrics by telling us to look at the Chern classes.

Before discussing more of the physics behind Calabi-Yau manifolds, we discuss another very important property, holonomy.

Theorem 2.4. *An n-dimensional Calabi-Yau manifold has $SU(d)$ holonomy, $d \leq n$.*

Proof. (sketch)
A CY is Kähler so we know that $Hol(CY_n) \subset U(n)$ so we just need to show that for any $h \in Hol(CY_n)$, $det(h) = 1$. As we know the holonomy is fundamentally related to the curvature as the curvature map spits out an infinitesimal rotation. So for $X^\mu \in T^{1,0}$,

$$\begin{aligned} X^\mu \mapsto & X^\mu\big(\delta^\nu_\mu + R^\nu{}_{\mu\sigma\bar{\lambda}}\epsilon^\sigma\bar{\delta}^{\bar{\lambda}}\big) = X^\mu h_\mu{}^\nu \\ \implies & det(h_\mu{}^\nu) = det(e^{R^\nu{}_{\mu\sigma\bar{\lambda}}\epsilon^\sigma\bar{\delta}^{\bar{\lambda}}}) = det(\delta^\nu_\mu + R^\nu{}_{\mu\sigma\bar{\lambda}}\epsilon^\sigma\bar{\delta}^{\bar{\lambda}}) \\ & = 1 + Tr(R^\nu{}_{\mu\sigma\bar{\lambda}}\epsilon^\sigma\bar{\delta}^{\bar{\lambda}}) + \ldots = 1 + \delta^\nu_\mu R^\nu{}_{\mu\sigma\bar{\lambda}}\epsilon^\sigma\bar{\delta}^{\bar{\lambda}} + \ldots \\ & = 1 + R^\mu{}_{\mu\sigma\bar{\lambda}}\epsilon^\sigma\bar{\delta}^{\bar{\lambda}} + \ldots = 1 + R_{\sigma\bar{\lambda}}\epsilon^\sigma\bar{\delta}^{\bar{\lambda}} + \ldots \\ & = 1 \end{aligned} \tag{2.25}$$

Thus, CY manifolds have holonomy, $h \in SU(d)$. □

So Calabi-Yau's have $SU(d)$ holonomy but we do not want the holonomy to be contained in $SU(d)$, $d < n$, we want it to fill out all of $SU(n)$. The reason is that a proper subgroup of $SU(n)$ will not allow chiral fermions, but our universe is chiral so it must fill out all of $SU(n)$ [8].

Let us see what else is so unique about Calabi-Yau manifolds and why they appear naturally. As discussed in [8], a superstring $M_4 \times M_6$ background must have an unbroken $\mathcal{N} = 1$ supersymmetry in the M_4 part which has maximal symmetry. This means that if a we have superspace translations generated by some spinor, (grassman odd) ε, then $0 = \delta_\varepsilon[\textit{bosonic field}] = [\textit{fermionic field}]$. For this to be true, variations of the fermionic fields must vanish. We can see the consequences of this by inspection of the 10-dimensional supergravity multiplet which includes a spin-$\frac{3}{2}$ field, ψ, a spin-$\frac{1}{2}$ field, λ, a scalar field, φ, and a 2-form potential B_{MN}. The field strength associated to B is labeled H. When M_{10} decomposes to $M_4 \times M_6$ the fields in the theory will also decompose. For example our spin-$\frac{3}{2}$ field becomes[11] $\psi_M = \psi_\mu \otimes \psi_m$. This means we can look at the superspace variations on each space separately. Taking a look at the variation of ψ we can make some comments about the background.

$$\delta_\varepsilon \psi_\mu = \nabla_\mu \varepsilon + \frac{\sqrt{2}}{32} e^{2\varphi}\Big(\gamma_\mu \gamma_5 \otimes H\Big)\varepsilon, \tag{2.26a}$$

$$\delta_\varepsilon \psi_m = \nabla_m \varepsilon + \frac{\sqrt{2}}{32} e^{2\varphi}\Big(\gamma_m H - 12 H_m\Big)\varepsilon, \tag{2.26b}$$

where $H = H_{\rho\sigma\delta}\gamma^{\rho\sigma\delta}$, and $H_m = H_{mqr}\gamma^{qr}$. Forcing equation 2.26a to vanish, combined with $\delta_\varepsilon \lambda = 0$ tells us that ε is killed by the field strength, $H\varepsilon = 0$. So, equation 2.26b reduces to

$$\tilde{\nabla}_m \varepsilon = (\nabla_m - \beta H_m)\varepsilon, \tag{2.27}$$

with $\beta = \frac{3}{8}\sqrt{2}e^{2\phi}$. Setting this to zero, for nontrivial translation, tells us that for $\varepsilon_{(10)} = \varepsilon_{(4)} \otimes \varepsilon_{(6)}$, and preservation of supersymmetry in M_4, $\varepsilon_{(6)}$ must be covariantly constant [6]. These constraints, and others, force M_6 to be a Kähler, Ricci flat manifold with $SU(3)$ holonomy[12]. Of course this is none other than our favorite Calabi-Yau manifold, and so, $M_6 \cong CY_3$.

We turn our attention now to the Hodge diamond of a Calabi-Yau. There is so much structure in these spaces that new symmetries appear. We can see this in Hodge diamond of CY manifolds. Just as we defined a special form on for Kähler manifolds we can do so for CY manifolds. We say the *Calabi-Yau form*, $\Omega \in \bigwedge^{n,0} T^*$, is a holomorphic top form on CY_n. We actually gain more from the CY form using the fact that Calabi-Yau manifolds have trivial canonical bundle [7]. We have then, $K_m \cong \bigwedge^{n,0} T^* \cong CY_3 \times \mathbb{C}$. Observe that the CY form, $\Omega \in K_m$, it is holomorphic, forms are also sections which are defined over an open subset of the domain, and that CY manifolds are connected means we can apply the maximum modulus principle applies here[13]. This tells us that the CY form, Ω, is really constant i.e. nowhere vanishing. Trivially then Ω is harmonic and therefore defines the equivalence class in the De Rahm cohomology group, $[\Omega] \in H^{3,0}(CY_3)$. Where $\Omega \sim f\Omega$ with f is a holomorphic function and therefore also constant by the maximum modulus principle. Thus Ω is unique up to constant rescaling. From this we have $h^{3,0} = 1$, and by Hodge duality we also get $h^{0,3} = 1$ [4, 5]. Because the canonical bundle is trivial such a form can always be found. We can now define the form:

Definition 2.30. The Calabi-Yau form, $\Omega \in \bigwedge^{n,0} T^*$, is a nowhere vanishing holomorphic top form on CY_n.

Furthermore, if we take any $\alpha \in H^{0,p}$ there is a unique $\beta \in H^{0,n-p}$, on CY_n, such that

$$\int_M \alpha \wedge \beta \wedge \Omega = 1. \tag{2.28}$$

[11] We follow notation from [8] where capital Latin letters for 10 dimensional space, Greek letters for the 4 dimensional space and lowercase Latin letters for the 6 dimensional space.

[12] For all the specific details consult [8].

[13] See theorem A.1 for a reminder of the principle.

Hence, the closed top $(3,3)$-form in 2.28 is not exact and therefore we have a duality between $H^{0,p}$ and $H^{0,n-p}$. This implies a new Hodge number relation: $h^{0,p} = h^{0,n-p}$. Of course we can then conjugate the space and we have $h^{p,0} = h^{n-p}$ [4, 5]. So, we see that the Hodge duality for De Rahm cohomology on a real manifold gives a 'decomposition' to a Hodge-like duality[14] for holomorphic and anti-holomorphic forms on a Calabi-Yau. This special feature of CY manifolds is known as **holomorphic duality** and is guaranteed by the existence and uniqueness of the CY form.

We can go even further to reduce the Hodge numbers using a theorem due to Bochner and considering how forms transform under the holonomy the $SU(n)$ holonomy of Calabi-Yau's.

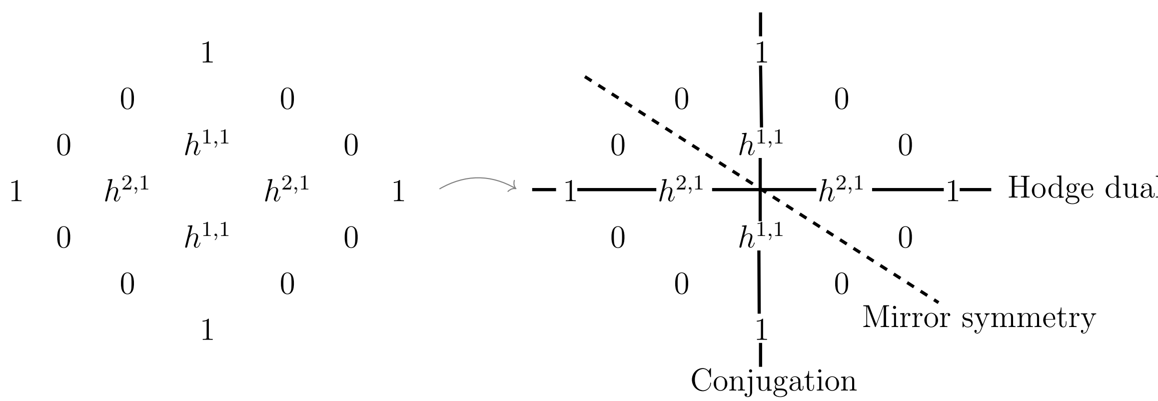
Theorem 2.5 (Bochner)**.** *Any harmonic s-form, ω, can be written as*

$$F(\omega) = R_m{}^n \omega_{nr_2\cdots r_s}\omega^{mr_2\cdots r_s} + \frac{1}{2}(s-1)R_m{}^n{}_p{}^q \omega_{nqr_3\cdots r_s}\omega^{mpr_3\cdots r_s}. \tag{2.29}$$

If $F(\omega) \geq 0$, then $\nabla\omega = 0$.

That is to say that the s-form ω transforms trivially under holonomy (given $F \geq 0$). If we take a 1-form on a CY, then $F(\omega) = 0$ as CY is Ricci flat, and hence $\nabla\omega = 0$. However, forms on a Calabi-Yau transform either the fundamentally representation or dual to the fundamental representation of the $SU(n)$ holonomy of CY. This can only be true if the space of 1-forms on CY is empty, $h^1 = 0$ [4, 5]. Of course then, by the consequences of Observation number 3 of a Kähler manifold, $h^{0,1} = 0 = h^{1,0}$, as the dimension is nonnegative.

So, we have the top and (Hodge dual) bottom of the Hodge diamond are 1's, as the space is connected. The right corner and (conjugate) left corner are also 1, due to definition 2.30. The remaining perimeter Hodge numbers are killed by theorem 2.5, conjugation, and holomorphic duality just leaving a nontrivial interior. However, our wonderful CY manifolds unsurprisingly relate these numbers with all the same dualities. The Hodge diamond (for a CY_3) is drawn below.



We have redrawn the Hodge diamond this time highlighting the symmetries[15]. We do not have this extra dashed axis of symmetry yet, but is worth saying a few words about now. It turns out that for every CY_d there is a mirror CY'_d in which $h^{1,1}$ on CY_d is equal to $h^{d-1,1}$ on CY'_d [2, 4, 5]. Once we talk about field theories, we will see the isomorphism set up between $H^{1,1}(CY_3)$ and $H^{2,1}(CY'_3)$ that gives mirror symmetry.

[14] We still have Hodge duality on CY.

[15] This graphic was inspired by [4].

Chapter 3

Topological Quantum Field Theory

When discussing quantum field theories (QFTs), it is taken for granted that the QFT is defined over a fixed space-time (usually Minkowski) and not much thought is given past this. However, picking a background in which the QFT exists can and will greatly affect predictions of the theory. It is possible, however, to consider QFTs that only depend on the *generic* global structure of the space on which they are defined (that is the topology) rather than the specific local structure (the metric of the specific manifold). These are the distinguished topological quantum field theories (TQFTs). Discussion of which will provide us with the tools necessary for mirror symmetry and topological strings.

In order to formalize TQFTs, we must start with the wide, wonderful, world of categories. Then we discuss TQFT from axioms which will give us a deep understanding for the last section of cohomological field theories (CoFT). This chapter follows M. Atiyah [11], for the axioms, and then closely follows [6] for the discussion of CoFTs.

3.1 Categories

Categories may feel like a big, scary monster but they are crucial to understanding what a TQFT truly is. Our aim here is not to become categorical experts but really to gain familiarity with that monster; break down that wall of mysticism, making clear that the monster is really our friend. Our discussion of categories is heavily motivated through hands-on examples.

Defining a category is not so straight forward as we first need to understand its constituents. These are things called objects and morphisms. We give a 'definition' of what a mathematical object is, but it only becomes clear through the following examples.

Definition 3.1. A (mathematical) object is a specific element of a type of structure.

This may be confusing, so we shall clarify what this means through the following examples.

Example 3.1. Consider sets. The set $S = \{\text{dog, cat, fish}\}$ is an specific set and therefore a mathematical object. Further examples of objects that are sets are $\mathbb{R}, \mathbb{C}, E = \{2n \mid n \in \mathbb{N}\}$. Now consider groups. The set of integers under addition, $(\mathbb{Z}, +)$ forms a group, $(\mathbb{Z}, +)$ is then an object. Further examples include $(\mathbb{R}, +)$, $(\mathbb{R} - \{0\}, \cdot)$, and $(SU(2), \times)$ where $\times$ is matrix multiplication. The list goes on.

Definition 3.2. A morphism is a structure-preserving map between two objects of the same type.

Example 3.2. Let A and B be sets, if $f : A \to B$ then f is a morphism. Let G, H be groups then a homomorphism $h : G \to H$ is a morphism. Let V, W be vector spaces a linear operator $A : V \to W$ is a morphism.

Consider a map ρ, from the group $(\mathbb{Z}, + \pmod 2)$ to the group $(\mathbb{Z}, + \pmod 3)$, such that $\rho(0) = 0$, $\rho(1) = 1$. This map is **not** a morphism as, $\rho(1 + 1 \pmod 2) = \rho(0) = 0$ but $\rho(1) + \rho(1) \pmod 3 = 2$, i.e. it is not a homomorphism.

Definition 3.3. A category, C, is a mathematical structure that satisfies the following axioms.

C0) There is a collection of objects $X, Y, Z \cdots$ that make up the objects of C, denoted $obj(C)$ and we say $X \in obj(C)$. There is also a collection of morphisms, $f, g, h \cdots$ between objects, denoted $mor(C)$, with $f \in mor(C)$. Note that $\forall\ X \in obj(C)\ \exists$ an identity morphism, $I_X : X \to X$, the 'do nothing' map.

C1) Composition of morphisms. There is a bilinear operation from the morphisms of C to the morphisms of C, $\circ : mor(C) \times mor(C) \to mor(C)$, such that for morphisms $f : X \to Y$, $g : Y \to Z$, and $h : Z \to W$,

C2) the composition is associative:

$$(h \circ g) \circ f = h \circ (g \circ f), \text{ and}$$

C3) there is an identity composition. That is $\forall\ f \in mor(C)$

$$I_Y \circ f = f = f \circ I_X$$

Remark. Every object in a category has an identity map just like how every set has the empty set as a subset. We get these identity morphisms on objects for free, as they are the do nothing map. Furthermore even though the definition of the identity morphism map did not explicitly require it to be the do nothing map, axiom (C3) does require it.

Example 3.3. Consider A set of numbers, $Nu = \{1, 2, 3\}$, a set of fruit, $Fr = \{\text{apple},\ \text{banana},\ \text{raspberry}\}$, a set of meals, $Me = \{\text{apple sauce},\ \text{smoothie},\ \text{jam}\}$, and finally a set of pets, $Pe = \{\text{dog},\ \text{cat},\ \text{fish}\}$. Let us also say that for each set, $Nu,\ Fr,\ Me,\ Pe$, we have an identity map on each which maps every element of ever set to itself: denoted $I_{Nu},\ I_{Fr},\ I_{Me},\ I_{Pe}$ respectively. Now define the functions $f : Nu \to Fr$ such that $f(1) = \text{apple},\ f(2) = \text{banana},\ f(3) = \text{raspberry}$, a list of groceries to get from the store, $g : Fr \to Me$ such that $g(\text{apple}) = \text{apple sauce},\ g(\text{banana}) = \text{smoothie},\ g(\text{raspberry}) = \text{jam}$, meals to make with your new groceries, and $h : Me \to Pe$ such that $h(\text{apple sauce}) = \text{dog},\ h(\text{smoothie}) = \text{cat}$, and $h(\text{jam}) = \text{fish}$, the meals for each of your pets. Let us check this is a category of sets.

C0) We have a collection of four sets: numbers, fruit, meals, and pets which are our objects. We also have relations between them; morphisms $f,\ g$, and h as the domains and codomains are of the same type (sets).

C1) Because $f : Nu \to Fr$, $g : Fr \to Me$, and $h : Me \to Pe$, we can compose them in the usual way to get the maps such as $j = g \circ f : Nu \to Me$, such that

C2) it's associative. Which we can check explicitly:

$$
\begin{aligned}
(h \circ g) \circ f(1) &= (h \circ g)(\text{apple}) = \text{dog} = h(\text{apple sauce}) = h \circ (g \circ f)(1),\\
(h \circ g) \circ f(2) &= (h \circ g)(\text{banana}) = \text{cat} = h(\text{smoothie}) = h \circ (g \circ f)(2),\\
(h \circ g) \circ f(3) &= (h \circ g)(\text{raspberry}) = \text{fish} = h(\text{jam}) = h \circ (g \circ f)(3).
\end{aligned}
$$

C3) There are identity morphisms. We have these by construction, they trivially satisfy

$$
\begin{aligned}
I_{Fr} \circ f &= f = f \circ I_{Nu},\\
I_{Me} \circ g &= g = g \circ I_{Fr},\\
I_{Pe} \circ h &= h = h \circ I_{Me}.
\end{aligned}
$$

Hence, all the category axioms are satisfied and we have our very first category. The category of chores. In set theory the focus on is on the objects themselves. Notice here the emphasis is on the relations (morphisms) between the objects (sets).

Example 3.4. Consider the group $(G, +)$. The group operation induce morphisms through conjugation. For each $g' \in G$ then we have conjugation map as $h : G \to G$ such that $g \mapsto h + g + h^{-1}\ \forall\ g \in G$ and for each $h \in G$. Recall morphisms are structure persevering. These conjugation maps are homomorphisms as $h(g_1+g_2) = h+g_1+g_2+h^{-1} = h+g_1+h^{-1}+h+g_2+h^{-1} = h(g_1)+h(g_2)$ (they inherit the structure directly from G). We have $e \in G$ and hence $e : G \to G$ is the identity morphism. We can compose these maps as $h'' \circ h'(g) = h''(h'+g+h'^{-1}) = h''+h'+g+h'^{-1}+h''^{-1} = (h''+h')(g)$, and note the composition gets the associative property directly from the fact it is a group. Thus, we have a category with a single object G.

We have not explicitly considered any the opposite conjugation with $h^{-1} \in G$, if we do then we get inverse morphisms. Directly following the composition of morphisms property from above we see, $h \circ h^{-1}(g) = (h + h^{-1})(g) = e(g) = g \ \forall \ g, h, h^{-1} \in G$. Since the homomorphisms are invertible we have isomorphisms. This defines a groupoid.

Example 3.5. Some further examples of categories (also found in [12]) include: abelian groups with homomorphisms, vector spaces with linear maps, and topological spaces with continuous maps.

Having a category is great but, as with most everything, the real power lies in relating multiple categories. Short of spoiling the punchline, this is the magic of TQFTs. We relate categories by a special types of functions called functors.

Definition 3.4. Let C, D be categories. Let $X, Y, Z \in obj(C)$ and $f : X \to Y,\ g : Y \to Z$ be morphisms, $f, g \in mor(C)$. A functor, F, is a map between categories, $F : C \to D$ such that

F1) objects are associated to objects, that is $F : X \mapsto F(X) \in obj(D)$,

F2) morphisms are associated to morphisms, that is $F : f \mapsto F(f) \in mor(D)$,

F3) each identity morphism in C is mapped to its corresponding identity morphisms in D, that is $\forall \ X \in obj(C),\ F : I_X \mapsto F\big(I_{F(X)}\big)$, and

F4) composition of morphisms is preserved, that is $F(g \circ f) = F(g) \circ F(f) \ \forall \ f, g \in mor(C)$.

The big idea is that we can map from one mathematical structure to another while maintaining the structure of each. Here are a few examples to help clarify.

Example 3.6. If C is a category we have the *identity functor*, $F : C \to C$ such that $\forall \ X \in obj(C) \ F(X) = X,\ F(I_X) = I_X$, and $\forall \ f \in mor(C),\ F(f) = f$. Trivially we have $F(g \circ f) = g \circ f = F(g) \circ F(f)$.

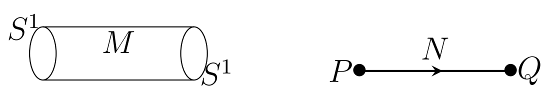
A slightly more exciting example: take the category of abelian groups, *aGroups* and the category of sets, *Sets*. We can define a functor $F : aGroups \to Sets$ by taking groups to the set over which they are defined. For example $(\mathbb{Z}, + \pmod 2)$, $(\mathbb{Z}, + \pmod 4)$ gets sent to $\mathbb{Z}_2 \cong \{0, 1\}$ and $\mathbb{Z}_4 \cong \{0, 1, 2, 3\}$ respectively. A homomorphism that sends 0 and 1 in $\mathbb{Z}_2$ to 0 and 2 in $\mathbb{Z}_4$ gets sent to those maps (forgetting about the modulo addition). This is an example of a *forgetful functor*, as we lost the structure that makes it a group and only have the sets and maps between them. [12].

Now that we have a feel for categories it is time to shift our focus to the categories necessary for TQFT. First, the category of Hilbert (vector) spaces, *Hilb*. Of course this should come as no surprise and we want to do quantum field theory at some point. A new category we need is the category of $(n+1)$-dimensional cobordisms, $nCob$. First, of course, the definition of an $(n+1)$-dimensional cobordism.

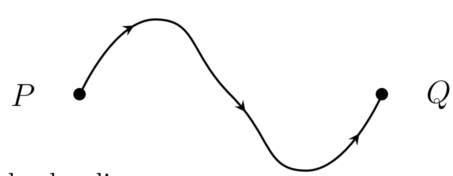
Definition 3.5. Let W_1, W_2 be n-dimensional manifolds. A <u>cobordism</u> is a compact $(n+1)$-dimensional manifold, M, such that the boundary is the disjoint union of the two n-dimensional manifolds, $\partial M = W_1 \coprod W_2$. We say W_1 and W_2 are <u>cobordant</u> if there exists a cobordism between them. If M is also oriented then it is an oriented cobordism[1].

One way to think about cobordisms is that they are 'maps' between two manifolds of the same dimension living in one higher dimension. Again we give several examples to make concrete the abstract definitions.

Example 3.7. The natural first image that comes to mind is a cylinder. Two circles, S^1, are cobordant by the surface of the cylinder, M. Of course these are 1 dimensional manifolds with a 2 dimensional connecting manifold; nothing is stopping us from going down (or up, although harder to draw) in dimension. Points P, Q here are cobordant because we can connect them with a directed[a] line, $N = \overline{PQ}$.



Just to be crystal clear, M is a cobordism, $\partial M = S^1 \coprod S^1$, N is also a cobordism, $\partial N = P \coprod Q$. These cobordisms are trivial because $W_1 \cong W_2$, and $M \cong W_1 \times I$, where I is some closed interval [13]. Note that the cobordism, N, above is equivalent to the following cobordism:

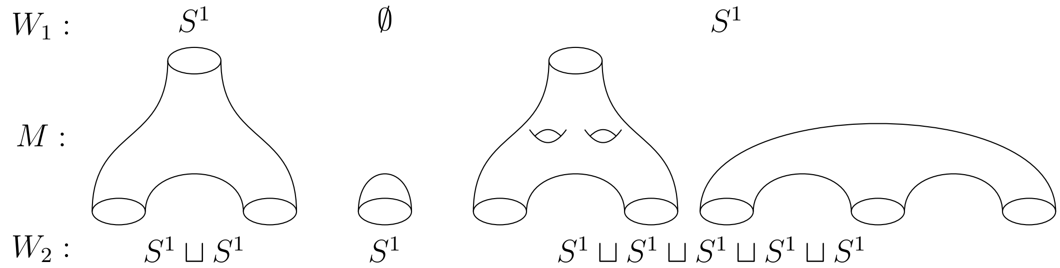


[a]Directed because it is an oriented cobordism.

The previous example was a cobordism between two identical manifolds. Naturally, the definition does not require this and we can consider cobordisms between different manifolds.

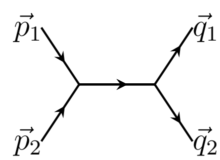
[1]We will only consider this case save for example 3.8

Example 3.8. As another example we can consider a cobordism connected different spaces (still n-dimensional manifolds). We draw a cobordisms, M, between manifold of circle(s), W_1, and different manifold (again of circles), W_2. Implicit in this are cobordisms between a manifold and the trivial manifold (the emptyset). Of course, M, does not have to be something simple like a cylinder it can be a manifold of any genus g.



Talking about cobordisms is awesome because it gives us a new way to think about particle interactions. Rather than strictly thinking about the mathematical objects and shapes we can consider physical examples.

Example 3.9. The path of a particle with boundary conditions, $x(0) = a,\ x(t) = b$, defines a 0-dimensional cobordism between the boundary states. Extending this to interactions, Feynman diagrams are cobordisms (in momentum space) between (initial) particles of momentum $\vec{p}_1,\ \vec{p}_2$, and (final) particles of momentum $\vec{q}_1,\ \vec{q}_2$.



Here the cobordism, M, is the quantum processes/propagation of particles freely propagating to interaction, virtual particle exchange, and free propagation from $W_1 = \{\vec{p}_1\} \cup \{\vec{p}_2\}$ to $W_2 = \{\vec{q}_1\} \cup \{\vec{q}_2\}$, which make up the boundary $\partial M = \{\vec{p}_1\} \cup \{\vec{p}_2\} \cup \{\vec{q}_1\} \cup \{\vec{q}_2\}$.

Example 3.9 brings up another important point. The above picture is just a single interaction, there are infinitely many other paths the particles could take and infinitely many other diagrams due to all of the quantum processes. Considering all the possible cobordisms and therefore a possible paths particles can take leads to the path integral approach to QFT. Moreover, this is just a 0-dimensional cobordism, and we can consider higher dimensional cases. If we take 1-dimensions for example we have string propagation, both open and closed. Cobordism M in example 3.7 is exactly this: A string propagating freely. The first cobordism in example 3.8 is the aptly named pair of pants string interaction.

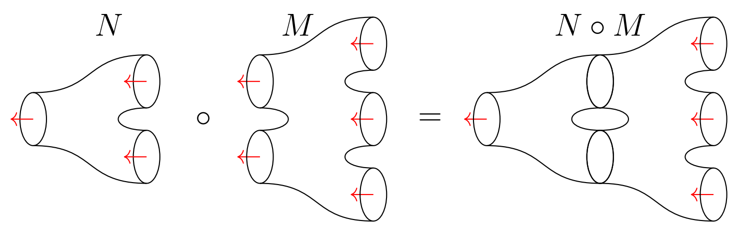
One final point before moving forward. Back at the start of the section we said there was a category of cobordisms, $nCob$. Now that we understand what a cobordism is let us verify we have a category.

Proposition 3.1. A collection of n-dimensional oriented manifolds and oriented $(n+1)$-dimensional cobordisms forms a category, $nCob$.

Solution. (sketch) Here we demonstrate the elements necessary for the category, rather than rigorously prove it is one.

C0) Our objects are n-dimensional oriented manifolds. This means for every $W \in obj(nCob)$ we have a $\bar{W}$ the same manifold but with opposite orientation. Then our morphisms are $(n+1)$-dimensional compact oriented manifold. Note that $\forall\ W \in obj(nCob)\ \exists$ a cobordism, $M : W \to W,\ s.t.\ M \cong W \times I$.

C1) Let W_1 and W_2 be cobordant through M, $\partial M = W_1 \coprod W_2$ and W_2 and W_3 be cobordant through N, $\partial N = W_2 \coprod W_3$. Through the orientation we can compose the cobordisms, $N \circ M : W_1 \to W_3$ since $\partial(N \circ M) = W_1 \coprod W_3$, so $N \circ M$ is a cobordism. For example,



The orientation (arrows) tell us how to compose cobordisms.

C2) It is associative. Thinking about the morphisms like legos, the orientation of each tells us how we can attach them, i.e only outgoing boundaries to incoming boundaries.

C3) The identity composition is the ability to tack-on a cylinder anywhere, as long as the cobordism N has at least one outgoing or one incoming boundary. Since the cylinder is the identity cobordism (morphism), we can arbitrarily grow or shrink the interval, I. So, if $N : W_1 \to W_2$, then $(W_2 \times I) \circ N = N = N \circ (W_1 \times I)$.

So we have a category of cobordisms, $nCob$ [13, 14]. □

An important point to stress here is the orientation. It was necessary to make $nCob$ a category and it tells us how to put cobordisms together. It also tells us how we can take them apart. It is possible to decompose cobordisms. Without loss of generality we can do the opposite of composition shown in (C1) above to break spaces apart.

We took the scenic route, but we are ready for topological quantum field theory. We have the necessary tools in hand to dissect TQFTs and learn the inner workings of the theory.

3.2 Topological Field Theory

As we discussed at the start of the chapter, the case and point of TQFTs are that they just depend on the topology not the local structure. However, we need a way to communicate the mathematical, topological properties with the physical. We do this by relating the structure of the Hilbert space of the quantum theory to the topological structure of n-dimensional cobordisms[2].

Definition 3.6. Let Σ be an n-dimensional, oriented, closed manifold (with opposite oriented manifold Σ^*), and M a $(n+1)$-dimensional cobordism, $\partial M = \Sigma_1 \sqcup \Sigma_2$. Further, Let Λ be a ring, over which is a quantum theory; a Hilbert space, $\mathcal{H}$. A topological quantum field theory is a functor, Z, from the category of n-dimensional cobordisms to the category of Hilbert vector spaces, $Z : nCob \to Hilb$

- TQFT0) Z is a functor so the functor axioms must be satisfied. Namely, each n-dimensional manifold, Σ, is associated to a Hilbert space, $Z(\Sigma) = \mathcal{H}$, and cobordisms are associated to linear maps, $Z(M) \in Hom(Z(\Sigma_1), Z(\Sigma_2))$ *i.e.* $Z(M) : Z(\Sigma_1) \to Z(\Sigma_2)$, equivalently $Z(M) : \mathcal{H}_1 \to \mathcal{H}_2$.

such that the following axioms are satisfied.

- TQFT1) Z is involutory, by which we mean the opposite oriented manifold is mapped to the dual Hilbert space associated to the original oriented manifold, $Z(\Sigma^*) = Z(\Sigma)^*$. If $Z(\Sigma) = \mathcal{H}$ then $Z(\Sigma^*) = \mathcal{H}^*$ the dual Hilbert space.
- TQFT2) We say Z is multiplicative, $Z(\Sigma_1 \sqcup \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2)$, meaning multiple disjoint manifolds are associated to a tensor product Hilbert space, $\mathcal{H}_1 \otimes \mathcal{H}_2$.
- TQFT3) The empty manifold, $\Sigma = \emptyset$, is associated to the ring which the theory is defined over, $Z(\emptyset) = \Lambda$.
- TQFT4) Lastly, we require the identity cobordism to associate to the identity of the Hilbert space, $Z(\Sigma \times I) = I_{Z(\Sigma)}$.

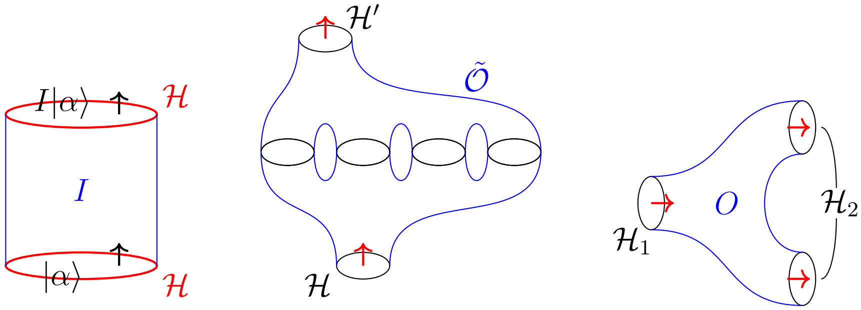
Following from[3] [11, 13, 15].

As a minimum requirement Λ, must be a ring. We will take $\Lambda = \mathbb{C}$ with the additional structure of a field over which we have a Hilbert space. This means our maps to or from empty sets correspond to maps to or from $\mathbb{C}$. So, the TQFT functor sets up an *equivalence* between space(-time) manifolds and Hilbert spaces of a quantum theory. To help illustrate what the TQFT functor does we give the following examples.

[2]For a reminder of a ring structure and a field structure see definitions A.8 and A.9.

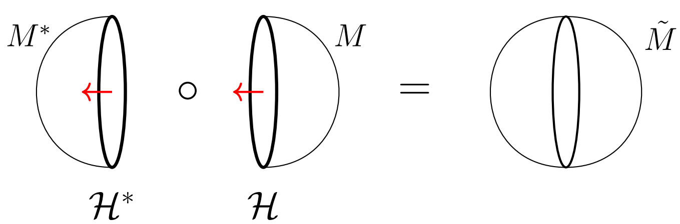
[3]There is a slight abuse of notation in the definition with I, however, it should be contextually clear whether we are discussing the interval, I, with identity cobordisms or the identity operator on the Hilbert space, $I_{Z(\Sigma\times I)} \in Hom(\Sigma, \Sigma)$.

Example 3.10. First, let us consider the trivial case of the identity cobordism (the cylinder). The incoming and outgoing boundaries are the same Hilbert space, $\mathcal{H}$, and the cobordism between them $(\Sigma \times I)$ is the identity operator, I. A state, $|\alpha\rangle \in \mathcal{H}$ is sent to $I|\alpha\rangle$. We can also consider arbitrary transformation from a Hilbert space, $\mathcal{H}$, to the transformed Hilbert space, $\mathcal{H}' = \tilde{\mathcal{O}}\mathcal{H}$. Or a map from one Hilbert space, $\mathcal{H}_1$ to a totally different Hilbert space $\mathcal{H}_2$, through operation O.



Following the lectures of [15], we can further dissect these axioms. We interpret the n-dimensional boundary manifolds as spaces of a quantum theory, a Hilbert space and the $(n+1)$-dimensional cobordisms (space-time manifolds) are simply the evolution operators as. For example, taking the second cobordism in example 3.10, $M \cong \tilde{\mathcal{O}}$ corresponds to a map $Z(M) : \mathcal{H} \to \mathcal{H}'$. Moreover, if the cobordism, M, is a closed manifold, we think of it as the correlation function. Let us piece this together.

First, the closed space-time manifolds are associated to complex numbers as they are maps from $\emptyset \to \emptyset$ which through the TQFT corresponds to $\mathbb{C} \to \mathbb{C}$. That is $Z(M) : \mathbb{C} \to \mathbb{C}$, which will be the path integral: $Z(M) = \int e^{-S[\varphi]} D\varphi$, for some field φ on M. Then, we take boundary states, (with an orientation) $|M\rangle \in \mathbb{H}$, to be cobordisms from the empty set to an n-dimensional manifold[4], $Z(M) : \mathbb{C} \to \mathcal{H}$. Just as we have the map from the field (empty manifold) to the Hilbert vector space (space manifold), we can have a map in the opposite direction through the dual space, $Z(M^*) : \mathcal{H}^* \to \mathbb{C}$. By our TQFT axioms, this opposite boundary state is the same manifold but with opposite orientation, $M^* \cong \langle M|$.



$$Z(\tilde{M}) = \langle M|M\rangle \tag{3.1}$$

Figure 3.1: Demonstrates how the composition of cobordisms of a closed manifold correspond to calculating expectation values. Where the ‘cap’ manifold corresponds to a boundary state, and $\tilde{M}$ is the total cobordism; the sphere.

[4]As in example 3.8.

Equation 3.1 is none other than the correlation function.

$$\begin{aligned} Z(\tilde{M}) &: \mathbb{C} \to \mathbb{C} \\ Z(\tilde{M}) &= \langle M^* M \rangle = \int_{\mathcal{H}} M^* M e^{-S[\varphi]} D\varphi \end{aligned} \tag{3.2}$$

The TQFT composition properties allow for manifolds (space-times) to be 'cut' and 'glued' together through these boundary states. We will discuss this further in the next section.

The math language of TQFTs has is strong; it told us how to think about quantum field theories in terms of manifolds. It also implies that the dynamics of QFT occur through the manifold topology changes. Transitioning to the physical language, our intuition tells us the dynamics due to topological properties is equivalent to a theory independent of a metric put on the space. However, the metrics are not completely removed from the story. As was alluded earlier, particles' worldsheets are cobordisms to which quantum theories are assigned via a TQFT functor[5]. The particles still must travel in some target space and we cannot avoid the dependence on the local structure here. Through the TQFT, however, we gain the ability investigate *only* the global structure (the topology of the worldsheet) [6]. So, equation 3.2, which was a result of the topology of the cobordism, should be independent of any metric on it. Then, we can recast the definition of a TQFT.

Definition 3.7. Let (M, g) be a smooth d-dimensional manifold with metric g. Let (Σ, h_{ab}) be a smooth 2-dimensional manifold with metric h_{ab} be a worldsheet embedded in M. Furthermore suppose we have a QFT defined over M with observables, $\mathcal{O}_i$, so we have $\langle \mathcal{O}_i \cdots \mathcal{O}_n \rangle_M$. We define the theory to be a topological quantum field theory if

$$\frac{\delta}{\delta h_{ab}} \big\langle \mathcal{O}_i \cdots \mathcal{O}_n \big\rangle_M = 0, \tag{3.3}$$

that is the correlator is independent of worldsheet metric.

There is some ambiguity in how to achieve this. For example one could try to explicitly build a theory by integrating over all worldsheet metrics h. However, as quantum gravity as not been figured out yet, this is quite difficult [6]. So, we will try a different method, first pioneered by E. Witten, cohomological field theories (CoFT) [16].

3.3 Cohomological Field Theories

As we know from Chapter 2, a the exterior derivative is a nilpotent operator[6] which gives rise to cohomology. In our field theories we do not have an exterior derivative yet, but if we had a nilpotent operator like the exterior derivative we could draw an analogy between the two and create a cohomology in the quantum theory. Our guiding questions for this section: How far does the analogy go? What insights are gained?

[5]For point particles: $0Cob$, for strings: $1Cob$ to which most examples have been drawn.

[6]A nilpotent operator, $\mathcal{O}$, satisfies $\mathcal{O}^n = 0$. In most all physical examples $n = 2$.

Example 3.11. We are already familiar with such operators from supersymmetry: grassman operators, the generator of superspace translations, Q, or a fermionic field, ψ.

With these ideas in mind, we now state what it means for a field theory to be cohomological, following from [6].

Definition 3.8. A cohomological field theory (also known as a TQFT of Witten type) is a quantum theory in which the following axioms are satisfied.

CoFT1) There is a nilpotent symmetry generator, $Q^2 = 0$.

CoFT2) The physical observables, $\mathcal{O}_i$, are Q-closed, that is $[Q, \mathcal{O}_i\} = 0$, and are independent of worldsheet metric, $\frac{\delta}{\delta h}\mathcal{O}_i = 0 \;\forall\; i$.

CoFT3) The vacuum state of the theory is symmetric, $\exists\; |0\rangle$ s.t. $Q|0\rangle = Q^\dagger|0\rangle = 0$.

CoFT4) Finally, the energy-momentum tensor is Q-exact, i.e. $T_{\alpha\beta} = \frac{\delta S}{\delta h^{\alpha\beta}} = \{Q, G_{\alpha\beta}\}$, for some operator $G_{\alpha\beta}$.

Of course the actual commutation relation will depend upon whether or not V obeys bosonic or fermionic statistics, but we use the anticommunitator for simplicity.

This lays the groundwork for interesting physics to come. In defining the cohomolgical field theory we have drawn an analogy between the exterior derivative, d, and a symmetry generator, Q. First, however, we should verify that these theories are indeed topological.

Proposition 3.2. A cohomological field theory is a topological field theory.

Solution. Suppose we have all the requirements for a cohomological field theory as defined above. Then consider

$$\begin{aligned} \frac{\delta}{\delta h}\langle \mathcal{O}_i \cdots \mathcal{O}_n\rangle &= \frac{\delta}{\delta h}\int \mathcal{O}_i \cdots \mathcal{O}_n e^{\frac{i}{\hbar}S[\varphi]} D\varphi \\ &= \frac{i}{\hbar}\int \mathcal{O}_i \cdots \mathcal{O}_n \frac{\delta S}{\delta h} e^{\frac{i}{\hbar}S[\varphi]} D\varphi \\ &= \frac{i}{\hbar}\int \mathcal{O}_i \cdots \mathcal{O}_n \{Q, G\} e^{\frac{i}{\hbar}S[\varphi]} D\varphi \\ &= \frac{i}{\hbar}\langle \mathcal{O}_i \cdots \mathcal{O}_n \{Q, G\}\rangle \\ &= (-1)^n \frac{i}{\hbar}\langle Q\mathcal{O}_i \cdots \mathcal{O}_n G\rangle + \frac{i}{\hbar}\langle \mathcal{O}_i \cdots \mathcal{O}_n G Q\rangle \\ &= 0 \end{aligned} \tag{3.4}$$

Thus, the correlator is independent of the worldsheet metric and the cohomolgical theory is a topological one [6]. □

We do not need to start by finding a Q-exact $T_{\alpha\beta}$, which may be very difficult to do. If given a Q-exact lagrangian, we can guarantee the theory will have a Q-exact energy-momentum tensor.

Proposition 3.3. If we start with a Q-exact lagrangian, $\mathcal{L} = \{Q, V\}$, then the theory will be cohomological and therefore topological.

Solution.

$$\begin{aligned} T = \frac{\delta S}{\delta h} &= \frac{\delta}{\delta h} \int \{Q, V\} \\ &= \int \left(Q \frac{\partial V}{\partial h} + \frac{\partial V}{\partial h} Q \right) \\ &= \left\{ Q, \int \frac{\partial V}{\partial h} \right\}. \end{aligned}$$

Thus, a Q-exact lagrangian gives rise to a Q-exact stress energy momentum tensor which implies the correlator is independent of the metric given the observables are metric independent. □

Quite trivially then, we also have,

Corollary. *If the action S is Q-exact, the theory is topological.*

There is a lot of power behind CoFTs. One reason being, path integrals (and so too correlators), which correspond to closed manifolds can be calculated exactly. In order to show this, consider a Q-exact lagrangian, but with an additional parameter, t altering the potential, $\mathcal{L} = \{\mathcal{Q}, tV\}$. Then the action is $S = \{\mathcal{Q}, t \int V\}$. Consider now the variation of the closed manifold (path integral) corresponding to this action,

$$\begin{aligned} \frac{d}{dt} \int e^{-\{\mathcal{Q}, tV\}} DX &= - \int \{\mathcal{Q}, V\} e^{-\{\mathcal{Q}, tV\}} DX \\ &= \langle \mathcal{Q} V + V \mathcal{Q} \rangle = 0, \end{aligned} \tag{3.5}$$

as our operator kills the vacuum. Since the physical operators are t independent,

$$\frac{d}{dt} \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \langle \mathcal{O}_1 \cdots \mathcal{O}_n \{\mathcal{Q}, V\} \rangle = 0. \tag{3.6}$$

We can then evaluate in the limit where $t \to \infty$. We only get contributions from terms which minimize the potential, V, i.e. classical solutions and our path integral becomes finite dimensional $\left(\int DX \to \int dx \right)$. Which implies we can evaluate path integrals exactly.

Further pushing the analogy between CoFTs and d-cohomology, we derive the so called decent equations. In d-cohomology we start with some scalar function, a 0-form, and take derivatives to obtain p-forms on M. Can we do the same in the CoFT? Yes, we can.

Proposition 3.4. Cohomological Field Theories on an m-dimensional manifold, M, admit physical p-forms, $(p \leq m)$.

Solution. To start, we first observe that the conserved quantities are:

$$H = \int T_{00} d^m x = \left\{ Q, \int G_{00} d^m x \right\}, \tag{3.7a}$$

$$P_\alpha = \int T_{\alpha 0} d^m x = \left\{ Q, \int G_{\alpha 0} d^m x \right\}. \tag{3.7b}$$

That is to say they are Q-exact. Consider then the momentum density[7], $\mathcal{P}_\alpha = \{Q, G_{\alpha 0}\}$, and a scalar observable (Q-closed) of the theory $\mathcal{O}^{(0)}$.

By taking an exterior derivative of a p-form we get a $(p+1)$-form. So, we should try to take a derivative (physically this corresponds to looking at the commutator of the momentum density operator) of the scalar operator and see what happens.

$$\begin{aligned}
\frac{d}{dx^\alpha}\mathcal{O}^{(0)} &= i\big[\mathcal{P}_\alpha, \mathcal{O}^{(0)}\big] \\
&= i[\{Q, G_\alpha\}, \mathcal{O}^{(0)}] \\
&= i[(QG_\alpha + G_\alpha Q), \mathcal{O}^{(0)}] \\
&= i[QG_\alpha, \mathcal{O}^{(0)}] + i[G_\alpha Q, \mathcal{O}^{(0)}] \\
&= iQ[G_\alpha, \mathcal{O}^{(0)}\} + i\cancelto{0}{\{Q, \mathcal{O}^{(0)}\}}G_\alpha + iG_\alpha\cancelto{0}{\{Q, \mathcal{O}^{(0)}\}} + i[G_\alpha, \mathcal{O}^{(0)}\}Q \\
&= Q\mathcal{O}^{(1)}_\alpha \pm \mathcal{O}^{(1)}_\alpha Q \\
&= [Q, \mathcal{O}^{(1)}_\alpha\}.
\end{aligned} \tag{3.8}$$

Where we have introduced a new operator, $\mathcal{O}^{(1)}$, by saying the fermionic operator, G_α, does not (anti-)commute trivially with $\mathcal{O}^{(0)}$,

$$\big[G_\alpha, \mathcal{O}^{(0)}\big\} = -i\mathcal{O}^{(1)}_\alpha. \tag{3.9}$$

For simplicity in writing, we will take $\mathcal{O}^{(i)}$ to be fermionic. We see this new operator, $\mathcal{O}^{(1)}_\alpha$, is a 1-form, and that the derivative of the scalar observable is Q-exact. One way to view this result is by the physicists trick of separating the derivative. We have

$$\begin{aligned}
\frac{d}{dx^\alpha}\mathcal{O}^{(0)} &= \{Q, \mathcal{O}^{(1)}_\alpha\} \\
\text{“} \implies \mathrm{d}\mathcal{O}^{(0)} &= \{Q, \mathcal{O}^{(1)}_\alpha\} dx^\alpha \text{”},
\end{aligned} \tag{3.10}$$

to which we define $\mathcal{O}^{(1)} = \mathcal{O}^{(1)}_\alpha dx^\alpha$. We have then $\mathrm{d}\mathcal{O}^{(0)} = \{Q, \mathcal{O}^{(1)}\}$.

Now, we need to show that $\mathcal{O}^{(1)}$ is physical. That is, we need to show $\mathcal{O}^{(1)}$ is Q-closed. In order to do so, consider a closed curve, $\gamma \subset M$.

$$\left\{ Q, \int_\gamma \mathcal{O}^{(1)} \right\} = \int_\gamma \{Q, \mathcal{O}^{(1)}\} = \underbrace{\int_\gamma \mathrm{d}\mathcal{O}^{(0)} = \int_{\partial\gamma} \mathcal{O}^{(0)}}_{\text{by Stoke's theorem}} = 0. \tag{3.11}$$

[7]We take G_α to be fermionic.

Where $\partial\gamma = 0$ since γ is closed. Consequently, the new operator, $\mathcal{O}^{(1)}$, is Q-closed and therefore[8] physical.

So, we started from a 0-form and went to a 1-form; repeating this process we obtain all p-forms over M. These are the so called decent equations:

$$\{Q, \mathcal{O}^{(0)}\} = 0, \tag{3.12a}$$

$$\{Q, \mathcal{O}^{(1)}\} = \mathrm{d}\mathcal{O}^{(0)}, \tag{3.12b}$$

$$\begin{aligned}\{Q, \mathcal{O}^{(2)}\} &= \mathrm{d}\mathcal{O}^{(1)}, \\ &\vdots\end{aligned} \tag{3.12c}$$

$$\{Q, \mathcal{O}^{(m)}\} = \mathrm{d}\mathcal{O}^{(m-1)}, \text{and finally,} \tag{3.12d}$$

$$\mathrm{d}\mathcal{O}^{(m)} = 0. \tag{3.12e}$$

This solution follows [6] closely with a few more details. □

Proposition 3.4 really drives home the connection between manifold cohomology and this new field theory cohomology. We are finding they are the same thing. To this end, the symmetry generator, Q, acts as (becomes) the exterior derivative operator, and the physical observables are the differential forms. Naturally, an equivalence class of observable can be made:

$$\mathcal{O}_a \sim \mathcal{O}_b \iff \mathcal{O}_a - \mathcal{O}_b = [Q, \Lambda\}. \tag{3.13}$$

Of course, equation 3.13 is the exact same equivalence class used to define the cohomology groups of a manifold. It should come as no surprise that this equivalence translates to the correlators.

Proposition 3.5. If $\mathcal{O} \sim \mathcal{O}'$, then $\langle \mathcal{O}_1 \cdots \mathcal{O} \cdots \mathcal{O}_n \rangle = \langle \mathcal{O}_1 \cdots \mathcal{O}' \cdots \mathcal{O}_n \rangle$.

Solution. Let $\mathcal{O} \sim \mathcal{O}'$, then $\mathcal{O} = \mathcal{O}' + [\mathcal{Q}, \Lambda\} \implies$

$$\begin{aligned}\langle \mathcal{O}_1 \cdots \mathcal{O} \cdots \mathcal{O}_n \rangle &= \langle \mathcal{O}_1 \cdots (\mathcal{O}' + [\mathcal{Q}, \Lambda\}) \cdots \mathcal{O}_n \rangle \\ &= \langle \mathcal{O}_1 \cdots \mathcal{O}' \cdots \mathcal{O}_n \rangle + \langle \mathcal{O}_1 \cdots \mathcal{O}_n [\mathcal{Q}, \Lambda\} \rangle \\ &= \langle \mathcal{O}_1 \cdots \mathcal{O}' \cdots \mathcal{O}_n \rangle,\end{aligned} \tag{3.14}$$

as $\mathcal{Q}$ annihilates the vacuum, independent of Λ's spin-statistics ($\{\ ,\ \}$ or $[\ ,\ ]$). □

3.3.1 Cobordisms and Cohomological Field Theory

We have discussed at length topological field theories on a manifold and how this gives rise to a cohomology. Let us not forget how crucial cobordisms are to the story. The composition of cobordisms gives us a way to 'cut' (and 'glue') manifolds. This is done by a clever insertion of the identity operator.

[8] It also will not depend on the metric as $\mathcal{O}^{(0)}$ did not.

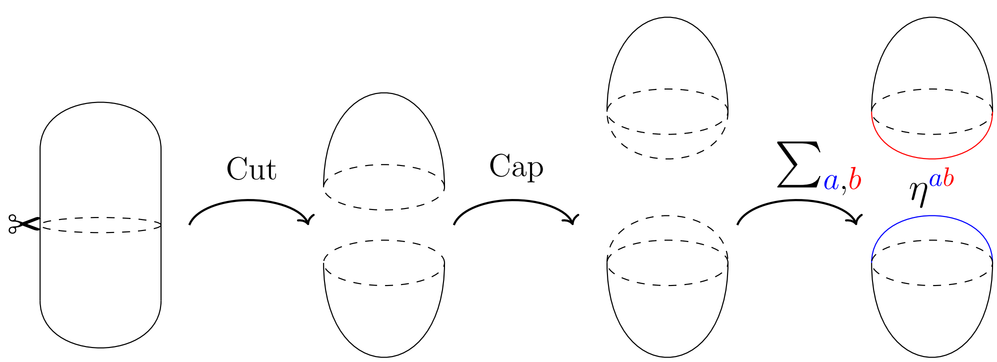
Recall from quantum mechanics, $I = \sum_{x\in\mathcal{H}} |x\rangle\langle x|$. This implies $I = \sum_{x,y} |x\rangle\langle x|y\rangle\langle y| = \sum_{x,y} |x\rangle\delta_{xy}\langle y|$, if states, x, y form an orthonormal basis of $\mathcal{H}$. Now, if the basis in not known to be orthonormal, we can instead replace δ_{xy} with some η^{xy}. Where we can think of this η as measuring the difference from the states being orthonormal.

Back to the matter at hand, we can use this to 'cut' closed manifolds in two ways. The first way is to cut the manifold such that the genus is preserved. When, a manifold is cut open, a boundary appears. The original manifold was closed an so should the resulting manifold after any cutting. So, the price to be paid is capping off the now open boundaries. Just one cannot be chosen though, all possible boundary states must be considered, and η accounts for the difference in the caps. So, given a closed manifold (some n-point correlation function) of some genus g, the cutting process is,

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_g = \sum_{a,b} \langle \mathcal{O}_1 \cdots \mathcal{O}_i \mathcal{O}_a \rangle_{g_1} \eta^{ab} \langle \mathcal{O}_b \mathcal{O}_{i+1} \cdots \mathcal{O}_n \rangle_{g_2}. \tag{3.15}$$

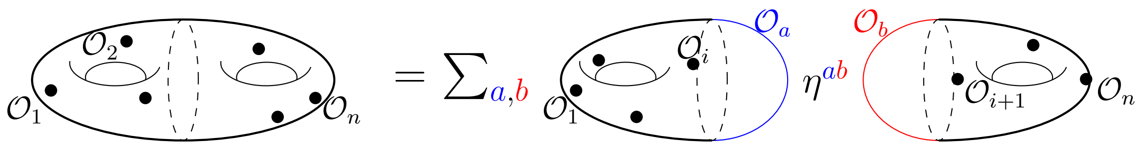
Where the sum is written explicitly for clarity. The manifold gets split into two manifolds each of lower genus g_1 and g_2 such that $g_1 + g_2 = g$. The operators $\mathcal{O}_a$, $\mathcal{O}_b$ correspond to $|x\rangle$, $\langle y|$ respectively. To help illustrate this, consider the following examples.

Example 3.12. The first cutting process demonstrated with the sphere as with figure 3.1. The sphere is a composition of two cobordisms. The first from $\mathbb{C}$ to some $\mathcal{H}$ and then from that $\mathcal{H}$ back to $\mathbb{C}$. By cutting the manifold open we have to put caps on the now exposed boundary states, then sum over all possible boundary states.



Of course the space-time manifold corresponds to operators, but we've omitted writing any above to help with conceptual understanding. In the case of the sphere, each hemisphere corresponded to a single operator. Given any arbitrary closed manifold, we can think of the operators as be 'insertion points' on the manifold. At the very least as a representation of as the area around the insertion point being the section of the cobordism in the decomposition corresponding the the labeled operator.

Example 3.13. Now, consider the case of $g = 2$ from equation 3.15.



$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_2 = \sum_{a,b} \langle \mathcal{O}_1 \cdots \mathcal{O}_i \mathcal{O}_a \rangle_1 \eta^{ab} \langle \mathcal{O}_b \mathcal{O}_{i+1} \cdots \mathcal{O}_n \rangle_1. \tag{3.16}$$

We now make a few observations about equation 3.15. First, the most a correlator can be reduced is to a product of 2-point functions. Since[9],

$$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle = \langle \mathcal{O}_1 \mathcal{O}_a \rangle \eta^{ab} \langle \mathcal{O}_b \mathcal{O}_2 \rangle. \tag{3.17}$$

So, the 2-point correlation function gives rise to this η. One way to view η^{ab} is as a 'metric like' operator; one that raises and lowers indices, enforcing contraction between $\mathcal{O}_a$ and $\mathcal{O}_b$. The 3-point functions are quite special. This is because we can decompose any n-point correlator in terms of products of the 3-point function, but also they determine the structure constants of the ring of operators. For now, we label,

$$c_{abc} = \langle \mathcal{O}_a \mathcal{O}_b \mathcal{O}_c \rangle_0, \tag{3.18}$$

and by equation 3.17 we have,

$$\mathcal{O}_a \mathcal{O}_b = c_{ab}{}^{c} \mathcal{O}_c. \tag{3.19}$$

Before we show that n-point correlation functions decompose into products of 3-point functions, there is one more topological changing action we can take.

The second way in which a manifold can be cut is buy cutting to a hole, in which case the genus is lowered by one. Another way to think about this process is by pinching a manifold at one of the holes and then drawing it off to infinity [6]. This process is done in a similar way to equation 3.15,

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_g = (-1)^F \eta^{ab} \sum_{a,b} \langle \mathcal{O}_a \mathcal{O}_b \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_{g-1} \tag{3.20}$$

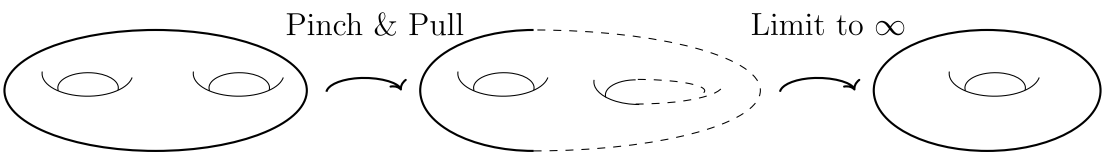
where

$$F = \begin{cases} 0 & \mathcal{O}_a,\ \mathcal{O}_b \text{ Bosonic (commute)} \\ 1 & \mathcal{O}_a,\ \mathcal{O}_b \text{ Fermionic (anticommute)} \end{cases}$$

Again the sum is written explicitly for clarity. The following examples are to help demonstrate this new cutting procedure.

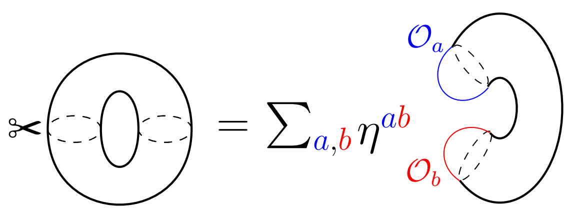
[9]This is pictorial shown in example 3.12

Example 3.14. This example illustrates the second possible way to cut a manifold; Cutting as a pinch and pull. The process for reducing a manifold's genus by one as described by [6]. Thinking of the delta function, if we say the 'area' of the hole must remain constant, in the limit as the manifold is stretched to infinity, the hole must collapse in one direction and expand out to infinity in the other. The whole becomes a 'line' in the manifold which we say is indistinguishable from the rest of the manifold[a].



[a]Like aligning two tables so flush the crack is not noticeable.

Example 3.15. The second cutting process, equation 3.20, demonstrated on a torus, $g = 1$. The process is very similar to equation 3.15 and example 3.13, where once the cut is made boundary states must be overlaid and account for the difference between them.



As was mentioned earlier, the 3-point function is special because any n-point correlator decomposes into products of the 3-point correlator. Let us now proof it.

Proposition 3.6. All n-point correlation functions on a manifold of some genus, g, can be reduced to products of 3-point functions on the sphere.

Solution. **Case 1)**: We start on the sphere, $g = 0$. Base case

$$\begin{aligned} \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle_0 &= \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_a \rangle_{g_1} \eta^{ab} \langle \mathcal{O}_b \mathcal{O}_3 \mathcal{O}_4 \rangle_{g_2}, \\ 0 &= g_1 + g_2, \\ \implies g_1 &= -g_2, \end{aligned} \tag{3.21}$$

but every $g \geq 0$, so $g_1 = 0 = g_2$. Assume true for n, that is,

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_0 = \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_a \rangle_0 \eta^{ab} \langle \mathcal{O}_b \mathcal{O}_3 \mathcal{O}_c \rangle_0 \eta^{cd} \cdots \eta^{x'y'} \langle \mathcal{O}_{y'} \mathcal{O}_{n-1} \mathcal{O}_n \rangle_0. \tag{3.22}$$

Now consider the $n+1$ case,

$$\begin{aligned}\langle\mathcal{O}_1\cdots\mathcal{O}_n\mathcal{O}_{n+1}\rangle_0 &= \langle\overbrace{\mathcal{O}_1\cdots\mathcal{O}_{n-1}\mathcal{O}_a}^{n\text{ operators}}\rangle_0\eta^{ab}\langle\mathcal{O}_b\mathcal{O}_n\mathcal{O}_{n+1}\rangle_0 \\ &= \langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_{a'}\rangle_0\eta^{a'b'}\cdots\eta^{x'y'}\langle\mathcal{O}_{y'}\mathcal{O}_{n-1}\mathcal{O}_a\rangle_0\eta^{ab}\langle\mathcal{O}_b\mathcal{O}_n\mathcal{O}_{n+1}\rangle_0,\end{aligned} \tag{3.23}$$

by the induction assumption. Hence, the n-point correlator on a sphere can be decomposed into a product of 3-point correlators.

Case 2): The 3-point function of a general manifold, $g \neq 0$. Our base case is $g = 1$.

$$\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle_1 = (-1)^F\eta^{ab}\langle\mathcal{O}_a\mathcal{O}_b\mathcal{O}_c\rangle_0\eta^{cd}\langle\mathcal{O}_d\mathcal{O}_1\mathcal{O}_e\rangle_0\eta^{ef}\langle\mathcal{O}_f\mathcal{O}_2\mathcal{O}_3\rangle_0, \tag{3.24}$$

by equation 3.22. Inducting on g, using equation 3.22, we assume,

$$\begin{aligned}\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle_g &= (-1)^{F\cdot g}\overbrace{\eta^{ab}\cdots\eta^{xy}}^{g\text{ times}}\langle\mathcal{O}_a\mathcal{O}_b\cdots\mathcal{O}_x\mathcal{O}_y\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle_0 \\ &= (-1)^{F\cdot g}\eta^{ab}\cdots\eta^{xy}\langle\mathcal{O}_a\mathcal{O}_b\mathcal{O}_\alpha\rangle_0\eta^{\alpha\beta}\cdots\eta^{\psi\omega}\langle\mathcal{O}_\omega\mathcal{O}_2\mathcal{O}_3\rangle_0.\end{aligned} \tag{3.25}$$

Now consider,

$$\begin{aligned}\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle_{g+1} &= (-1)^F\eta^{a'b'}\langle\mathcal{O}_{a'}\mathcal{O}_{b'}\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle_g \\ &= (-1)^F\eta^{a'b'}\langle\mathcal{O}_{a'}\mathcal{O}_{b'}\mathcal{O}_{\alpha'}\rangle_{g_1}\eta^{\alpha'\beta'}\langle\mathcal{O}_{\beta'}\mathcal{O}_1\mathcal{O}_{\gamma'}\rangle_{g_2}\eta^{\gamma'\delta'}\langle\mathcal{O}_{\delta'}\mathcal{O}_2\mathcal{O}_3\rangle_{g_3} \\ &= (-1)^{F\cdot(g_1+g_2+g_3)}(-1)^F\eta^{ab}\cdots\eta^{xy}\eta^{a'b'}\langle\mathcal{O}_{a'}\mathcal{O}_{b'}\mathcal{O}_\alpha\rangle_0\eta^{\alpha\beta}\cdots\eta^{\psi\omega}\langle\mathcal{O}_\omega\mathcal{O}_{b'}\mathcal{O}_{\alpha'}\rangle_0\eta^{\alpha'\beta'} \\ &\quad\times\eta^{\tilde{a}\tilde{b}}\cdots\eta^{\tilde{x}\tilde{y}}\langle\mathcal{O}_{\tilde{a}}\mathcal{O}_{\tilde{b}}\mathcal{O}_{\tilde{\alpha}}\rangle_0\eta^{\tilde{\alpha}\tilde{\beta}}\cdots\eta^{\tilde{\psi}\tilde{\omega}}\langle\mathcal{O}_{\tilde{\omega}}\mathcal{O}_1\mathcal{O}_{\gamma'}\rangle_0\eta^{\gamma'\delta'} \\ &\quad\times\eta^{AB}\cdots\eta^{XY}\langle\mathcal{O}_A\mathcal{O}_B\mathcal{O}_\aleph\rangle_0\eta^{\aleph\beth}\cdots\eta^{\gimel\daleth}\langle\mathcal{O}_\gimel\mathcal{O}_2\mathcal{O}_3\rangle_0\end{aligned} \tag{3.26}$$

by extensive use of the induction assumption. Hence, the 3-point correlator on any closed manifold can be written as a product of 3-point functions on the sphere.

At this point we have that any n-point function on the sphere reduces to a product of 3-point functions, and that the 3-point function on any closed manifold becomes a product of 3-point functions on the sphere. We now must induct on n for a surface of any genus g.

Case 3): Any correlator on any closed manifold. The base case of $n = 3$ is done in equation 3.25 in case (2). So, our induction assumption is,

$$\begin{aligned}\langle\mathcal{O}_1\cdots\mathcal{O}_n\rangle_g &= (-1)^{F\cdot g}\eta^{ab}\cdots\eta^{xy}\langle\mathcal{O}_a\mathcal{O}_b\cdots\mathcal{O}_x\mathcal{O}_y\mathcal{O}_1\cdots\mathcal{O}_n\rangle_0 \\ &= (-1)^{F\cdot g}\eta^{ab}\cdots\eta^{xy}\langle\mathcal{O}_a\mathcal{O}_b\mathcal{O}_{\alpha'}\rangle_0\eta^{\alpha'\beta'}\cdots\eta^{\psi'\omega'}\langle\mathcal{O}_{\omega'}\mathcal{O}_{n-1}\mathcal{O}_n\rangle_0.\end{aligned} \tag{3.27}$$

In order to obtain the desired result, putting the results of case (1) and (2) together we have,

$$\begin{aligned}\langle\mathcal{O}_1\cdots\mathcal{O}_n\mathcal{O}_{n+1}\rangle_g &= \langle\mathcal{O}_1\cdots\mathcal{O}_{n-1}\mathcal{O}_{a'}\rangle_{g_1}\eta^{a'b'}\langle\mathcal{O}_{b'}\mathcal{O}_n\mathcal{O}_{n+1}\rangle_{g_2} \\ &= (-1)^{F\cdot g_1}\eta^{ab}\cdots\eta^{xy}\langle\mathcal{O}_a\mathcal{O}_b\mathcal{O}_{\alpha'}\rangle_0\eta^{\alpha'\beta'}\cdots\eta^{\psi'\omega'}\langle\mathcal{O}_{\omega'}\mathcal{O}_{n-1}\mathcal{O}_{a'}\rangle_0\eta^{a'b'} \\ &\quad\times(-1)^{F\cdot g_2}\eta^{AB}\cdots\eta^{XY}\langle\mathcal{O}_A\mathcal{O}_B\mathcal{O}_\Gamma\rangle_0\eta^{\Gamma\Delta}\cdots\eta^{\Psi\Omega}\langle\mathcal{O}_\Omega\mathcal{O}_{n-1}\mathcal{O}_n\rangle_0.\end{aligned} \tag{3.28}$$

Therefore, any n-point correlation function on any closed manifold can be reduced to calculating 3-point functions on the sphere. $\square$

In conclusion, topological field theories have been introduced rigorously by an axiomatic approach, and the physical interpretation of them, introduced by Witten [16] as cohomological field theory, has been investigated at length. In turn, our journey through CoFTs has laid the groundwork for our continued investigation of mirror symmetry and on to topological string theories, where we will see the consequence of these theories pan out.

Chapter 4

$\mathcal{N} = (2,2)$ Supersymmetry

Having viewed cohomological field theories in all their glory, we now want to see where they show up. That is we want to find a theory in which we can show it is cohomological and therefore topological. We will do so with the $\mathcal{N} = (2,2)$ supersymmetric theory. This is a special theory because it gives rise to mirror symmetry through its topological nature (Chapter 5).

We begin with a quick refresher on supersymmetry through describing the $\mathcal{N} = (2,2)$ supersymmetry algebra. The main sources for this chapter are [6, 9].

4.1 The Supersymmetry Algebra and Chiral Fields

Let us turn our focus now to string worldsheet manifolds, with complex coordinates $z, \bar{z}$ (we note the derivatives then as ∂_+ for ∂_z and ∂_- for $\partial_{\bar{z}}$). On the worldsheet there are time translations, space translations, and Lorentz rotations. Each generated by the Hamiltonian, momentum operator, and Lorentz Noether charge, H, P, and M respectively. Together H, P, M make up the Poincaré Algebra.

$$\begin{aligned} H &= -i(\partial_+ - \partial_-) \qquad & [M,H] &= -2P \\ P &= -i(\partial_+ + \partial_-) \qquad & [M,P] &= -2H \\ M &= 2z\partial_+ - 2\bar{z}\partial_- \qquad & [H,P] &= 0 \end{aligned} \tag{4.1}$$

However, fermions should also be in the picture and so we must expand the Poincaré algebra. This means there should be other possible translations corresponding to fermionic degrees of freedom. So, we arrive at *superspace* which consists of[1] two bosonic coordinates, and four fermionic coordinates,

$$\begin{gathered} z,\ \bar{z},\ \theta^+,\ \theta^-,\ \bar{\theta}^+,\ \bar{\theta}^- \\ [z,\bar{z}] = 0 \\ \left\{\theta^\alpha, \theta^\beta\right\} = \left\{\bar{\theta}^\alpha, \bar{\theta}^\beta\right\} = \left\{\theta^\alpha, \bar{\theta}^\beta\right\} = 0. \end{gathered} \tag{4.2}$$

[1]In the $\mathcal{N} = (2,2)$ case.

The indices α, β note the chirality and take on the values of $\pm$, and is the charge under Lorentz transformations[2].

Remark. By extending the space, integrals include superspace variables as well. We remind ourselves of Berezin integration. The fermionic nature of the variables implies that integration over them is **equivalent** to differentiation.

$$\int c\theta d\theta = c = \frac{\partial}{\partial\theta}[c\theta] \qquad \int c\ d\theta = 0 = \frac{\partial}{\partial\theta}[c], \tag{4.3}$$

for some constant c. Another way to think about Berezin integration is that it picks out the components corresponding to the measure. For example, $\int f(\theta, \bar{\theta})d\bar{\theta}$, picks out the terms in f with $\bar{\theta}$. Finally, we denote the integral over superspace by

$$\int [\quad]d^2zd^4\theta \equiv \int [\quad]dzd\bar{z}d\theta^+d\theta^-d\bar{\theta}^+d\bar{\theta}^-.$$

Just as translations in the bosonic coordinates was generated by some symmetry so are translations in the fermionic coordinates. The possible generators and supersymmetric derivatives are

$$\begin{aligned} Q_\pm &= \frac{\partial}{\partial\theta^\pm} + i\bar{\theta}^+\partial_\pm \qquad & D_\pm &= \frac{\partial}{\partial\theta^\pm} - i\bar{\theta}^\pm\partial_\pm \\ \bar{Q}^\pm &= -\frac{\partial}{\partial\bar{\theta}^\pm} - i\theta^\pm\partial_\pm \qquad & \bar{D}_\pm &= -\frac{\partial}{\partial\bar{\theta}^\pm} + i\theta^\pm\partial_\pm. \end{aligned} \tag{4.4}$$

Note that these generators are fermionic and satisfy the following anti-commutation relations.

$$\{Q_\pm, \bar{Q}_\pm\} = -2i\partial_\pm \tag{4.5a}$$

$$\{D_\pm, \bar{D}_\pm\} = 2i\partial_\pm \tag{4.5b}$$

$$\{Q_\pm, D_\pm\} = \{Q_\pm, \bar{D}_\pm\} = 0, \tag{4.5c}$$

and the other two have been omitted as they are just the conjugation of the last line in equation 4.5c.

We can define fields on superspace in the following way.

Definition 4.1. A field over superspace, known as a <u>superfield</u>, is a field $\mathcal{F}(z, \bar{z}, \theta^+, \theta^-, \bar{\theta}^+, \bar{\theta}^-)$ is obtained by performing superspace translations on a field $\varphi(z, \bar{z})$,

$$\begin{aligned} \mathcal{F}(z, \bar{z}, \theta^+, \theta^-, \bar{\theta}^+, \bar{\theta}^-) = \varphi &+ \theta^+\varphi_+ + \theta^-\varphi_- + \bar{\theta}^+\varphi_{\bar{+}} + \bar{\theta}^-\varphi_{\bar{-}} \\ &+ \theta^+\theta^-\varphi_{+-} + \theta^+\bar{\theta}^+\varphi_{+\bar{+}} + \theta^+\bar{\theta}^-\varphi_{+\bar{-}} + \theta^-\bar{\theta}^+\varphi_{-\bar{+}} + \theta^-\bar{\theta}^-\varphi_{-\bar{-}} + \bar{\theta}^+\bar{\theta}^-\varphi_{\bar{+}\bar{-}} \\ &+ \theta^+\theta^-\bar{\theta}^+\varphi_{+-\bar{+}} + \theta^+\theta^-\bar{\theta}^-\varphi_{+-\bar{-}} + \bar{\theta}^+\bar{\theta}^-\theta^+\varphi_{\bar{+}\bar{-}+} + \bar{\theta}^+\bar{\theta}^-\theta^-\varphi_{\bar{+}\bar{-}-} \\ &+ \theta^+\theta^-\bar{\theta}^+\bar{\theta}^-\varphi_{+-\bar{+}\bar{-}}. \end{aligned} \tag{4.6}$$

The expansion is finite due to the fermionic nature of the superspace variables.

The full 16 term superfield is a lot to work with and actually possesses an $\mathcal{N} = (4, 4)$ supersymmetry. In order to simplify this theory we introduce chiral and anti-chiral superfields.

[2]More will be discussed about this in the section 4.3.

Definition 4.2. A chiral superfield, Φ, is a superfield such that both,

$$\bar{D}_{\pm}\Phi = D_{\pm}\bar{\Phi} = 0, \tag{4.7}$$

are satisfied. The chiral constraint greatly reduces the general superfield, $\mathcal{F}$ to become

$$\begin{aligned}\Phi = \varphi - i\theta^{+}\bar{\theta}^{+}\partial_{+}\varphi - i\theta^{-}\bar{\theta}^{-}\partial_{-}\varphi - \theta^{+}\theta^{-}\bar{\theta}^{-}\bar{\theta}^{+}\partial_{+}\partial_{-}\varphi + \theta^{+}\psi_{+}\\ - i\theta^{+}\theta^{-}\bar{\theta}^{-}\partial_{-}\psi_{+} + \theta^{-}\psi_{-} - i\theta^{-}\theta^{+}\bar{\theta}^{+}\partial_{+}\psi_{-} + \theta^{+}\theta^{-}F.\end{aligned} \tag{4.8}$$

This superfield consists of a scalar boson, φ, a spin-$\frac{1}{2}$, Weyl fermion, ψ, and a spin-1 boson, F.

An anti-chiral superfield is given by the conjugation of a chiral superfield,

$$D_{\pm}\Phi = \bar{D}_{\pm}\bar{\Phi} = 0. \tag{4.9}$$

Again both equations must be satisfied to be anti-chiral.

Together the chiral, anti-chiral constraints on $\mathcal{F}$ reduce the supersymmetry to $\mathcal{N} = (2,2)$ as a Q transformed chiral superfield is still chiral.

$$\bar{D}_{\pm}\varepsilon^{\pm}Q_{\pm}\Phi = \varepsilon^{\pm}Q_{\pm}\bar{D}_{\pm}\Phi = 0, \tag{4.10}$$

Note that because $\varepsilon^{\pm}$ is some fermionic parameter, the factor $\varepsilon^{\pm}Q_{\pm}$ is bosonic. The same is true for $\bar{Q}$.

4.2 Nonlinear σ-Model

Recall that the symmetry is on a string worldsheet. By asking the worldsheet to have this $\mathcal{N} = (2,2)$ supersymmetry, the string target space is forced to have much structure. Specifically, the target space must be Kähler. Showing this is a tedious exercise in using the chain and Leibniz rule for grassman numbers. We only include a few pieces of the calculation for brevity. The original article by B. Zumino ([17]) is very instructive, we follow [6, 9, 17], [18, 19] also serve as useful references.

We, suggestively, write a function of n chiral superfields as[3] $K\left(\Phi^{i}, \bar{\Phi}^{\bar{i}}\right)$. Terms involving K over superspace are automatically invariant under supersymmetry transformations. The reason being that the Berezin integral picks out the highest component. Then, only the $\partial_{\pm}$ term of a Q (or $\bar{Q}$) transformation survives which is a total derivative.

$$\begin{aligned}\mathcal{L}_D = \int K d^4\theta = \tilde{F}(z,\bar{z}) \subset K\left(\Phi^{i}, \bar{\Phi}^{\bar{i}}\right)\\ \implies \delta_Q\mathcal{L}_D = \int \varepsilon^{\pm}Q_{\pm}K d^4\theta\\ \propto \partial_{\pm}\tilde{F}(z,\bar{z})\end{aligned} \tag{4.11}$$

[3] $\left(i \in \{1, \cdots, n\}\right)$.

for some epsilon superspace shift, and we have omitted the integration over $z, \bar{z}$ for clarity of the Berezin integral. Such terms in the Lagrangian are known as D-terms. Since D-terms are invariant, they should appear in the Lagrangian.

Now, the D-term action amounts to performing superspace derivatives on the function K. We do not immediately have this from the action, but we want to use the chiral nature of the fields to our advantage. To demonstrate this we give the following example.

Example 4.1.

$$\begin{aligned}\int f(\theta^+)d\theta^+ d^2z &= \int \frac{\partial}{\partial\theta^+}[f]d^2z \\ &= \int (D_+ + i\bar{\theta}^+\partial_+)fd^2z\end{aligned} \tag{4.12}$$

Note that the second term is a total derivative so we effectively term the integral into a superspace derivative.

Before moving on the determining the lagrangian we note the following derivatives needed in the calculation.

$$\begin{aligned}\bar{D}_\pm K\big(\Phi^i, \bar{\Phi}^{\bar{i}}\big) &= \Big(-\frac{\partial}{\partial\bar{\theta}^\pm} + i\theta^\pm\partial_\pm\Big)K\big(\Phi^i, \bar{\Phi}^{\bar{i}}\big) \\ &= -\frac{\partial K}{\partial\Phi^i}\frac{\partial\Phi^i}{\partial\bar{\theta}^\pm} - \frac{\partial K}{\partial\bar{\Phi}^{\bar{i}}}\frac{\partial\bar{\Phi}^{\bar{i}}}{\partial\bar{\theta}^\pm} + i\theta^\pm\Big(\frac{\partial K}{\partial\Phi^i}\partial_\pm\Phi^i + \frac{\partial K}{\partial\bar{\Phi}^{\bar{i}}}\partial_\pm\bar{\Phi}^{\bar{i}}\Big) \\ &= \frac{\partial K}{\partial\Phi^i}\Big(-\frac{\partial\Phi^i}{\partial\bar{\theta}^\pm} + i\theta^\pm\partial_\pm\Phi^i\Big) + \frac{\partial K}{\partial\bar{\Phi}^{\bar{i}}}\Big(-\frac{\partial\bar{\Phi}^{\bar{i}}}{\partial\bar{\theta}^\pm} + i\theta^\pm\partial_\pm\bar{\Phi}^{\bar{i}}\Big) \\ &= \frac{\partial K}{\partial\Phi^i}\bar{D}_\pm\Phi^i + \frac{\partial K}{\partial\bar{\Phi}^{\bar{i}}}\bar{D}_\pm\bar{\Phi}^{\bar{i}} \\ &= \frac{\partial K}{\partial\bar{\Phi}^{\bar{i}}}\bar{D}_\pm\bar{\Phi}^{\bar{i}}.\end{aligned} \tag{4.13}$$

Where getting to the last line we have used the fact that Φ^i is chiral to eliminate the first term. Likewise for the derivative's conjugate,

$$\begin{aligned}D_\pm K\big(\Phi^i, \bar{\Phi}^{\bar{i}}\big) &= \Big(\frac{\partial}{\partial\theta^\pm} - i\bar{\theta}^\pm\partial_\pm\Big)K\big(\Phi^i, \bar{\Phi}^{\bar{i}}\big) \\ &= \frac{\partial K}{\partial\Phi^i}\frac{\partial\Phi^i}{\partial\theta^\pm} + \frac{\partial K}{\partial\bar{\Phi}^{\bar{i}}}\frac{\partial\bar{\Phi}^{\bar{i}}}{\partial\theta^\pm} - i\bar{\theta}^\pm\Big(\frac{\partial K}{\partial\Phi^i}\partial_\pm\Phi^i + \frac{\partial K}{\partial\bar{\Phi}^{\bar{i}}}\partial_\pm\bar{\Phi}^{\bar{i}}\Big) \\ &= \frac{\partial K}{\partial\Phi^i}D_\pm\Phi^i.\end{aligned} \tag{4.14}$$

Recall from Chapter 2, that if the partial derivatives of a function[4] $K(A^i, \bar{A}^{\bar{j}})$ exist, then we can obtain a Kähler metric, $g_{i\bar{j}} = \frac{\partial^2 K}{\partial A^i \partial\bar{A}^{\bar{j}}}$. Naturally, a connection and curvature tensor can be defined

[4]A real function.

(equations 2.22, 2.23). The action over all superspace reduces as

$$
\begin{aligned}
S_D &= \int K\big(\Phi^i, \bar{\Phi}^{\bar{i}}\big) d^4\theta d^2 z, \\
&= \int D_+ D_- \bar{D}_+ \bar{D}_- K\big(\Phi^i, \bar{\Phi}^{\bar{i}}\big) d^2 z.
\end{aligned}
\tag{4.15}
$$

and we arrive at the Lagrangian

$$
\begin{aligned}
\mathcal{L} = &- g_{i\bar{j}} \partial^\alpha \varphi^i \partial_\alpha \bar{\varphi}^{\bar{j}} - 2i g_{i\bar{j}} \bar{\psi}_-^{\bar{j}} \mathcal{D}_+ \psi_-^i - 2i g_{i\bar{j}} \bar{\psi}_+^{\bar{j}} \mathcal{D}_- \psi_+^i - R_{i\bar{j}k\bar{l}} \psi_+^i \psi_-^k \bar{\psi}_+^{\bar{j}} \bar{\psi}_-^{\bar{l}} \\
&+ g_{i\bar{j}} F^i \bar{F}^{\bar{j}} - g_{i\bar{j}} F^i \Gamma^{\bar{j}}_{\bar{k}\bar{l}} \bar{\psi}_-^{\bar{k}} \bar{\psi}_+^{\bar{j}} - g_{i\bar{j}} \Gamma^i_{jk} \psi_+^j \psi_-^k \bar{F}^{\bar{j}} + g_{i\bar{j}} \Gamma^i_{jk} \psi_+^j \psi_-^k \Gamma^{\bar{j}}_{\bar{k}\bar{l}} \bar{\psi}_-^{\bar{k}} \bar{\psi}_+^{\bar{j}},
\end{aligned}
\tag{4.16}
$$

where we have introduced a covariant derivative,

$$
\mathcal{D}_\pm = \partial_\pm + \Gamma^i_{jk} \partial_\pm \varphi^j. \tag{4.17}
$$

Note that the field F has no kinetic term and so is not dynamic, they are *auxiliary* fields and a we can integrate them out by,

$$
\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial \bar{F}^{\bar{j}}} = g_{i\bar{j}} \big(F^i - \Gamma^i_{jk} \psi_+^j \psi_-^k\big), \\
\implies F^i &= \Gamma^i_{jk} \psi_+^j \psi_-^k,
\end{aligned}
\tag{4.18a}
$$

$$
\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial F^i} = g_{i\bar{j}} \big(\bar{F}^{\bar{j}} - \Gamma^{\bar{i}}_{\bar{j}\bar{k}} \bar{\psi}_+^{\bar{j}} \bar{\psi}_-^{\bar{k}}\big), \\
\implies \bar{F}^{\bar{j}} &= \Gamma^{\bar{j}}_{\bar{k}\bar{l}} \bar{\psi}_-^{\bar{k}} \bar{\psi}_+^{\bar{l}}.
\end{aligned}
\tag{4.18b}
$$

The Lagrangian simplifies nicely into,

$$
\mathcal{L} = -g_{i\bar{j}} \partial^\alpha \varphi^i \partial_\alpha \bar{\varphi}^{\bar{j}} - 2i g_{i\bar{j}} \bar{\psi}_-^{\bar{j}} \mathcal{D}_+ \psi_-^i - 2i g_{i\bar{j}} \bar{\psi}_+^{\bar{j}} \mathcal{D}_- \psi_+^i - R_{i\bar{j}k\bar{l}} \psi_+^i \psi_-^k \bar{\psi}_+^{\bar{j}} \bar{\psi}_-^{\bar{l}}. \tag{4.19}
$$

This is known as the non-linear σ-model (nlσm). We can see the fields of the string worldsheet propagate in a target space with metric, $g_{i\bar{j}}$, a covariant derivative dependent on the connection, Γ^i_{jk}, and a curvature tensor, $R_{i\bar{j}k\bar{l}}$, all of which correspond to a Kähler manifold. The target space of an $\mathcal{N} = (2,2)$ superstring theory is Kähler.

In the nlσm the scalar field, φ, on the worldsheet, Σ, takes on the role of the coordinates on the target space manifold, M. Thus, an embedding map of the string worldsheet into the target space $\varphi : \Sigma \hookrightarrow M$. Consequently, fermions on the worldsheet, $\psi, \bar{\psi}$ are spinor vector fields over the target space given by the pullback of the tangent bundle, $\varphi^* TM$ [9].

$$
\psi_\pm \in \Gamma(\varphi^*(T^{(1,0)}) \otimes S_\pm), \tag{4.20a}
$$

$$
\bar{\psi}_\pm \in \Gamma(\varphi^*(T^{(0,1)}) \otimes S_\pm). \tag{4.20b}
$$

This is neatly summarized in the following table.

Interpretation of Fields

Field	Σ	M
$\varphi, \bar{\varphi}$	C^∞-function	coordinate
ψ_+	(0,1)-fermion	$TM^{(1,0)}$
ψ_-	(1,0)-fermion	$TM^{(1,0)}$
$\bar{\psi}_+$	(0,1)-fermion	$TM^{(0,1)}$
$\bar{\psi}_-$	(1,0)-fermion	$TM^{(0,1)}$

4.3 R-symmetry

Now, we want to understand how the supersymmetry algebra fits in with the Poincaré algebra, specifically the Lorentz rotations. So, let us focus now on the Lorentz subgroup of Poincaré. The operators H, P are unaffected by the extension to superspace but the Lorentz generator gets additional terms,

$$M = 2z\partial_+ - 2\bar{z}\partial_- + \theta^+\frac{\partial}{\partial\theta^+} - \theta^-\frac{\partial}{\partial\theta^-} + \bar{\theta}^+\frac{\partial}{\partial\bar{\theta}^+} - \bar{\theta}^-\frac{\partial}{\partial\bar{\theta}^-}. \tag{4.21}$$

As the worldsheet is two dimensional the Lorentz rotation group is just $SO(2) \cong U(1)$, and so a charge, q, can be assigned for each Lorentz rotation. This means in the 2d theory, under a Lorentz transformation in which, $z \mapsto e^{iq}z$ which means then

$$\theta^\pm \mapsto e^{\pm i\frac{q}{2}}\theta^\pm, \quad \bar{\theta}^\pm \mapsto e^{\pm i\frac{q}{2}}\bar{\theta}^\pm. \tag{4.22}$$

The extra factor of $\frac{1}{2}$ is reflective of the fermionic nature of the variables. Then, in relation to the Poincaré algebra, the supersymmetric generators satisfy,

$$[M, Q_\pm] = \mp Q_\pm \tag{4.23a}$$

$$[M, D_\pm] = \mp D_\pm, \tag{4.23b}$$

conjugation of equations 4.23a and 4.23b gives the rest.

Along with being supersymmetric, the action obtained from equation 4.19 may possesses other symmetries as well. Namely there is a $U(1)_V$ vector symmetry and a $U(1)_A$ axial symmetry. The symmetry means the action is invariant under the passive coordinate transformations

$$U(1)_V\text{: } \theta^\pm \mapsto e^{-i\alpha}\theta^\pm \implies \bar{\theta}^\pm \mapsto e^{i\alpha}\bar{\theta}^\pm, \text{ and} \tag{4.24a}$$

$$U(1)_A\text{: } \theta^\pm \mapsto e^{\mp i\beta}\theta^\pm \implies \bar{\theta}^\pm \mapsto e^{\pm i\beta}\bar{\theta}^\pm. \tag{4.24b}$$

A chiral superfield will be invariant under the $U(1)_V$ rotation if the component fields' rotation charges are

$$\varphi : \; q_V = 0, \tag{4.25a}$$

$$\psi_\pm : \; q_V = -1, \text{ and} \tag{4.25b}$$

$$F : \; q_V = 2. \tag{4.25c}$$

So, the chiral superfields transforms under $U(1)_V$ as

$$\Phi^i(z,\theta^\pm,\bar\theta^\pm) \mapsto e^{i\alpha q_V^i}\Phi^i(z,e^{-i\alpha}\theta^\pm,e^{i\alpha}\bar\theta^\pm). \tag{4.26}$$

Then Φ^i will also be invariant under the $U(1)_A$ if the component fields rotation charges are

$$\varphi\ ,\ F:\ q_A = 0, \tag{4.27a}$$
$$\psi_+:\ q_A = 1,\ \text{and} \tag{4.27b}$$
$$\psi_-:\ q_A = -1. \tag{4.27c}$$

In which case the Φ^i transforms in a similar way as before

$$\Phi^i(z,\theta^\pm,\bar\theta^\pm) \mapsto e^{i\beta q_A^i}\Phi^i(z,e^{\mp i\beta}\theta^\pm,e^{\pm i\beta}\bar\theta^\pm). \tag{4.28}$$

Together $U(1)_V \times U(1)_A$ is the R-symmetry group. The R-symmetries are generated by

$$F_V = -\theta^+\frac{\partial}{\partial\theta^+} - \theta^-\frac{\partial}{\partial\theta^-} + \bar\theta^+\frac{\partial}{\partial\bar\theta^+} + \bar\theta^-\frac{\partial}{\partial\bar\theta^-}, \tag{4.29a}$$
$$F_A = -\theta^+\frac{\partial}{\partial\theta^+} + \theta^-\frac{\partial}{\partial\theta^-} + \bar\theta^+\frac{\partial}{\partial\bar\theta^+} - \bar\theta^-\frac{\partial}{\partial\bar\theta^-}. \tag{4.29b}$$

In relation to the supersymmetry algebra we have,

$$\begin{aligned} &[F_V,Q_\pm] = Q_\pm && [F_A,Q_\pm] = \pm Q_\pm \\ &[F_V,\bar Q_\pm] = -\bar Q_\pm && [F_A,\bar Q_\pm] = \mp\bar Q_\pm. \end{aligned} \tag{4.30}$$

4.4 Vanishing Anomolies

Whenever we have a symmetry in the classical theory we must be cautious during the transition to the quantum theory. Symmetries in the classical theory that do not make it to the quantum theory are *anomalous*. So, let us check if the R-symmetries are anomalous for the path integral built from the nlσm lagrangian 4.19. The partition function will look like

$$\begin{aligned}\int e^{-S}D\varphi^iD\psi_+^iD\psi_-^iD\bar\varphi^{\bar i}D\bar\psi_+^{\bar i}D\bar\psi_-^{\bar i} = \int \exp\bigg(\int g_{i\bar j}\partial^\alpha\varphi^i\partial_\alpha\bar\varphi^{\bar j} + 2ig_{i\bar j}\bar\psi_-^{\bar j}\mathcal{D}_+\psi_-^i + 2ig_{i\bar j}\bar\psi_+^{\bar j}\mathcal{D}_-\psi_+^i \\ + R_{i\bar jk\bar l}\psi_+^i\psi_-^k\bar\psi_+^{\bar j}\bar\psi_-^{\bar l}d^2z\bigg)D\varphi^iD\psi_+^iD\psi_-^iD\bar\varphi^{\bar i}D\bar\psi_+^{\bar i}D\bar\psi_-^{\bar i}.\end{aligned} \tag{4.31}$$

For the symmetries to survive, we need the measure to be invariant. Right away we have that the integrals over both $D\varphi$ and $D\bar\varphi$ are invariant as they have zero charge in both the vector and axial symmetries. Moreover, F is of no concern here as we have already eliminated it from the path integral by putting it on-shell. So, what about the ψ^i and $\bar{\psi}^{\bar{i}}$? In order to determine if their measure is invariant, first recall that the following path integral over a field, χ, with some operator, O, reduces as,

$$\begin{aligned}\int e^{\chi O \chi} D\chi &= \int \exp(\chi, O\chi) D\chi \\ &= \sqrt{\det(O)} = \prod_{i=1}^{\infty} \sqrt{\lambda_i}.\end{aligned} \tag{4.32}$$

Where the λ_i are the eigenvalues of the operator O. Now, since all the ψ (and the conjugates) are fermionic we have to pay attention to any zero modes. This is demonstrated in the following example.

Example 4.2. We begin by rewriting the path integral in the following way,

$$\int [\quad] D\chi \equiv \prod_i \int [\quad] d\chi^i, \tag{4.33}$$

for some field χ. Consider some action $S[\chi, \psi]$, including a term such as $ig_{i\bar{j}}\chi O\psi$ for some operator O. Where ψ is fermionic and ψ_0 is a 0 mode of O. For simplicity let us say it is the only zero mode. Then the partition function becomes

$$Z = \int e^{-S[\chi,\psi]} D\chi D\psi = \prod_i \int e^{-(\cdots + ig_{i\bar{j}}\chi O\psi + \dots)} d\psi^i D\chi = \int \prod_{i\neq 0} \int e^{-S} d\psi_0 d\psi_i D\chi = 0. \tag{4.34}$$

Because the Berezin integral zero modes kill the entire path integral which is bad. In order to solve this problem we remove the zero modes by hand. The procedure to do so is referred to as *absorption of zero modes* [6]. First, to demonstrate how this works consider the following example.

Example 4.3. Taking the same problem as example 4.2, but this time consider the correlator,

$$\langle\psi\rangle = \int \psi e^{-S} D\chi D\psi = \prod_i \int \chi_i e^{-(\cdots + ig_{i\bar{j}}\chi O\psi + \dots)} d\psi_i D\chi = \int \psi_0 \prod_{i\neq 0} \int e^{-S} d\psi_0 d\psi_i D\chi \neq 0. \tag{4.35}$$

Hence, inserting the zero modes in the correlator will preserve the path integral from vanishing.

Ok, so how do we know how many zero modes there are in order to add them all to the correlator? This is answered by the operator's *index*. Functions, f, that are sent to 0 by an operator, O, live in the kernel of the operator, $f \in \ker(O)$. The number of fields sent to 0 is then the dimension of the kernel. There is a caveat to the zero modes however. The operator in question here is the covariant derivative, $\mathcal{D}_\pm = \partial_\pm + \Gamma^i_{jk}\partial_\pm\phi^j$, which depends on background fields, namely the Kähler metric. Because the operators $\mathcal{D}_\pm$ have background field dependence, the eigenvalues will vary and therefore so will the number of zero modes. However, by considering the operator's adjoint, $\mathcal{D}^\dagger_\pm$, which has corresponding zero modes, the difference between zero eigenvalues of $\mathcal{D}$ and $\mathcal{D}^\dagger$ will remain constant:

$$k_\pm = \dim(\ker(\mathcal{D}_\mp)) - \dim(\ker(\mathcal{D}^\dagger_\mp)). \tag{4.36}$$

The integers k_+ and k_- are the operators' ($\mathcal{D}_-$ and $\mathcal{D}_+$ respectively) index. The two indexes are related as $k_- = -k_+$ via conjugation [6].

As was just mentioned, because $k_\pm$ is only dependent on background field parameters, it is independent of the string worldsheet embedding in the target space. That is to say it is a topological invariant. This leads to the very useful result by Atiyah and Singer.

Theorem 4.1 (Atiyah-Singer Index Theorem)**.** *Given a string embedding $\varphi : \Sigma \hookrightarrow M$, the index of an operator can be found by it's first Chern class $c_1(M)$ by*

$$k_- = \int_{\varphi(\Sigma)} c_1(M). \tag{4.37}$$

So, assuming $k_- > 0$, giving us k zero modes, we can add them in the path integral by the absorption of zero modes procedure.

$$\langle g_{i_1\bar{j}_1}\psi_-^{i_1}\bar{\psi}_+^{\bar{j}_1}\cdots g_{i_k\bar{j}_k}\psi_-^{i_k}\bar{\psi}_+^{\bar{j}_k}\rangle = \int g_{i_1\bar{j}_1}\psi_-^{i_1}\bar{\psi}_+^{\bar{j}_1}\cdots g_{i_k\bar{j}_k}\psi_-^{i_k}\bar{\psi}_+^{\bar{j}_k}e^{-\int \mathcal{L}d^2z}D\varphi^i D\psi_+^i D\psi_-^i D\bar{\varphi}^{\bar{i}} D\bar{\psi}_+^{\bar{i}} D\bar{\psi}_-^{\bar{i}}, \tag{4.38}$$

and now we check how the R-symmetries behave. The correlation function is invariant under an α $U(1)_V$ rotation as,

$$\begin{aligned}\langle g_{i_1\bar{j}_1}\psi_-^{i_1}\bar{\psi}_+^{\bar{j}_1}\cdots g_{i_k\bar{j}_k}\psi_-^{i_k}\bar{\psi}_+^{\bar{j}_k}\rangle &\xmapsto{U(1)_V} \langle g_{i_1\bar{j}_1}e^{i\alpha}\psi_-^{i_1}e^{-i\alpha}\bar{\psi}_+^{\bar{j}_1}\cdots g_{i_k\bar{j}_k}e^{i\alpha}\psi_-^{i_k}e^{-i\alpha}\bar{\psi}_+^{\bar{j}_k}\rangle \\ &= \langle g_{i_1\bar{j}_1}\psi_-^{i_1}\bar{\psi}_+^{\bar{j}_1}\cdots g_{i_k\bar{j}_k}\psi_-^{i_k}\bar{\psi}_+^{\bar{j}_k}\rangle.\end{aligned} \tag{4.39}$$

However, under a $U(1)_A$ rotation, β, the correlation function becomes

$$\begin{aligned}\langle g_{i_1\bar{j}_1}\psi_-^{i_1}\bar{\psi}_+^{\bar{j}_1}\cdots g_{i_k\bar{j}_k}\psi_-^{i_k}\bar{\psi}_+^{\bar{j}_k}\rangle &\xmapsto{U(1)_A} \langle g_{i\bar{j}}e^{-i\beta}\psi_-^{i_1}e^{-i\beta}\bar{\psi}_+^{\bar{j}_1}\cdots + g_{i_k\bar{j}_k}e^{-i\beta}\psi_-^{i_k}e^{-i\beta}\bar{\psi}_+^{\bar{j}_k}\rangle \\ &= e^{-2ik\beta}\langle g_{i_1\bar{j}_1}\psi_-^{i_1}\bar{\psi}_+^{\bar{j}_1}\cdots g_{i_k\bar{j}_k}\psi_-^{i_k}\bar{\psi}_+^{\bar{j}_k}\rangle.\end{aligned} \tag{4.40}$$

In order for the $U(1)_A$ symmetry and therefore the $U(1)_V \times U(1)_A$ R-symmetry group to be preserved in the quantum theory (i.e. not anomalous) $k_\pm = 0$ by inspection of equation 4.40. As a consequence, the target space must have vanishing first Chern class, by the Atiyah-Singer index theorem. This is really wonderful because we have seen vanishing $c_1(M)$ before, specifically for Calabi-Yau manifolds. So, if the target space of our string embedding is not just Kähler, as it was for free in the nlσm, but rather Calabi-Yau, we have vanishing anomalies!

In summary, the luxury of $\mathcal{N} = (2,2)$ supersymmetry on the string worldsheet means the string target space is Kähler. The lowest component field, φ, can be viewed as coordinates in the target space providing the string embedding. The spinor fields $\psi_\pm$ (and their conjugates) take on the role of vector fields in the tangent and cotangent bundles. We find that the target space will be Calabi-Yau if the Axial R-symmetry anomaly vanishes. Remember CY manifolds have mirror pairs and if we restrict ourselves to CY manifolds we will find a correspondence between theories on mirror pairs. In order to see this we must *twist* the theory.

Chapter 5

Twisting the Nonlinear σ Model

We are finally ready to see how the $\mathcal{N} = (2, 2)$ supersymmetric theory leads to a cohomological theory. We saw from the previous chapter a nilpotent operator which is the first important step. The $Q_\pm$, $\bar{Q}_\pm$ alone will not be enough for a topological theory. However, taking a special combination of the four will lead to two different cohomological theories called the A and B model through a topological twist. This chapter follows [6, 9] closely.

5.1 Twisting

A priori we have four nilpotent operators, $Q_\pm, \bar{Q}_\pm$ as possible contenders for a cohomological theory. Remember, however, we need our physical observables to be closed under the nilpotent operator. Notice from equations 4.1 and 4.5a, we can build H and P from Q anti-commutators.

$$H = \frac{1}{2}(\{Q_+, \bar{Q}_+\} - \{Q_-, \bar{Q}_-\}), \tag{5.1a}$$

$$P = \frac{1}{2}(\{Q_+, \bar{Q}_+\} + \{Q_-, \bar{Q}_-\}). \tag{5.1b}$$

However, we also have,

$$\begin{aligned} \{\bar{Q}_+ + Q_-, Q_+ - \bar{Q}_-\} &= \{\bar{Q}_+, Q_+\} - \{\bar{Q}_+, \bar{Q}_-\} + \{Q_-, Q_+\} - \{Q_-, \bar{Q}_-\} \\ &= -2i\partial_+ + 2i\partial_- \\ &= 2H, \end{aligned} \tag{5.2}$$

and the same can be done for P using $(Q_+ + Q_-)$. Also taking the combination $\bar{Q}_+ + \bar{Q}_-$ we have,

$$\{\bar{Q}_+ + Q_-, Q_+ - \bar{Q}_-\} = 2H = \{\bar{Q}_+ + \bar{Q}_-, Q_+ - \bar{Q}_-\}, \tag{5.3a}$$

$$\{\bar{Q}_+ + Q_-, Q_+ + \bar{Q}_-\} = 2P = \{\bar{Q}_+ + \bar{Q}_-, Q_+ + Q_-\}. \tag{5.3b}$$

In constructing H and P from $(\bar{Q}_+ + Q_-)$ and $(\bar{Q}_+ + \bar{Q}_-)$ we may have found a way in which they can be 'Q-exact'. This of course would imply they are 'Q-closed', which is exactly what we need of our observables in a cohomological theory. There is one caveat that we must address before moving forward.

Recall in the nlσm, the target space could have arbitrary curvature but there was a hidden assumption that the worldsheet was flat; that it had a flat metric. Normally this would be fine, but our goal is to show the correlators of this theory are independent of any worldsheet metric. This amounts to asking the supersymmetric variation to act in the following way, $\delta\Phi^i = \epsilon^+ Q_+ \Phi^i$. Originally, a superspace variation would transform the component fields as

$$\delta\varphi^i = \epsilon_+ \psi_-^i - \epsilon_- \psi_+^i, \tag{5.4a}$$
$$\delta\bar{\varphi}^{\bar{i}} = -\bar{\epsilon}_+ \bar{\psi}_-^{\bar{i}} + \bar{\epsilon}_- \bar{\psi}_+^{\bar{i}}, \tag{5.4b}$$
$$\delta\psi_+^i = 2i\bar{\epsilon}_- \partial_+ \varphi^i + \epsilon_+ \Gamma^i_{jk} \psi_+^j \psi_-^k \tag{5.4c}$$
$$\delta\bar{\psi}_+^{\bar{i}} = -2i\epsilon_- \partial_+ \bar{\varphi}^{\bar{i}} + \bar{\epsilon}_+ \Gamma^{\bar{i}}_{\bar{j}\bar{k}} \bar{\psi}_-^{\bar{k}} \bar{\psi}_+^{\bar{j}} \tag{5.4d}$$
$$\delta\psi_-^i = -2i\bar{\epsilon}_+ \partial_- \varphi^i + \epsilon_- \Gamma^i_{jk} \psi_+^j \psi_-^k \tag{5.4e}$$
$$\delta\bar{\psi}_-^{\bar{i}} = 2i\epsilon_+ \partial_- \bar{\varphi}^{\bar{i}} + \bar{\epsilon}_- \Gamma^{\bar{i}}_{\bar{j}\bar{k}} \bar{\psi}_-^{\bar{j}} \bar{\psi}_+^{\bar{k}}. \tag{5.4f}$$

We are saying we want ϵQ to be some scalar parameter, while keeping $Q^2 = 0$. That is to say we want some of the fermion operators, $Q_\pm, \bar{Q}_\pm$, and their ϵ parameters, to become scalars on Σ. For the parameters, $\epsilon^\pm$ (and their conjugates) to be a scalars we need

$$\nabla^{(\text{on }\Sigma)}\epsilon = \frac{\partial}{\partial x}\epsilon = 0. \tag{5.5}$$

In making this true, we are twisting the bundle the operators live in.

Recall from example 2.4, that bundles can twist over the base space. This twisting can be thought of as a measure of obstruction from being the trivial bundle. In the case of the string worldsheet, the trivial bundle is $\Sigma \times \mathbb{C}$. The fields' spin is determined by its $U(1)$ Lorentz charge, q_M, which can be viewed as the fields spin over a manifold and will correspond to a twisted bundle.

Example 5.1. Scalar fields, such as φ in equation 4.19, have $q_M = 0$ and therefore φ lives in the trivial worldsheet bundle $\Sigma \times \mathbb{C}$. However, the $\psi_\pm$ and $\bar{\psi}_\pm$ fields in equation 4.19 are spinor fields and take on values from $S_\pm$; not the trivial bundle.

So the act of twisting somehow alters the Lorentz $U(1)$ symmetry of the worldsheet. By picking a global $U(1)_R$ symmetry and combine it with Lorentz symmetry, $U(1)_M$, we have

$$U(1)_{\text{twist}} \subset U(1)_M \times U(1)_R, \tag{5.6}$$

such that some of $Q_\pm, \bar{Q}_\pm$ have charge 0. Now, this new $U(1)$ twisted symmetry means

$$\nabla\epsilon \longrightarrow D\epsilon = \frac{\partial}{\partial x}\epsilon + \Gamma\epsilon + A\epsilon, \tag{5.7}$$

where Γ is the spin-connection. Then, enforce by hand $A = -\Gamma$, so $\nabla\epsilon = 0$. Now that we have $\frac{\partial}{\partial x}\epsilon = 0$, we know ϵ is constant and without loss of generality we can pick $\epsilon = 1$.

Observe that the R-symmetry was not specified, this means we can choose either the vector or axial $U(1)_R$ symmetry. Twisting with the vector symmetry is the A-model and twisting with the axial is the B-model. We start with the A-model before taking a look at the B-model.

5.2 The A-model

Take $U(1)_R$ to be $U(1)_V$ in equation 5.6. The twisted Lorentz generator, M_A, is defined to be

$$M_A = M - F_V, \tag{5.8}$$

so the new charge, $q_{M_A} \in U(1)_{\text{twist}}$, is given by,

$$q_{M_A} = q_M - q_V. \tag{5.9}$$

Then, we note the commutation relations of the twisted Lorentz generator, M_A with the supercharges.

$$[M_A, Q_+] = -2Q_+ \tag{5.10a}$$
$$[M_A, Q_-] = 0 \tag{5.10b}$$
$$[M_A, \bar{Q}_+] = 0 \tag{5.10c}$$
$$[M_A, \bar{Q}_-] = 2\bar{Q}_- \tag{5.10d}$$

So, we pick the A-twisted operator as

$$Q_A = \bar{Q}_+ + Q_-. \tag{5.11}$$

Proposition 5.1. The A-twisted supercharge, Q_A is nilpotent.

Solution. This is simple enough to see as,

$$\begin{aligned} Q_A^2 &= (\bar{Q}_+ + Q_-)(\bar{Q}_+ + Q_-) \\ &= \bar{Q}_+^2 + \bar{Q}_+ Q_- + Q_- \bar{Q}_+ + Q_-^2 \\ &= \{\bar{Q}_+, Q_-\} = 0. \end{aligned} \tag{5.12}$$

Hence, Q_A is nilpotent. □

Now, taking a look back at equation 5.3a-f, we see that the Hamiltonian and momentum operators are actually Q_A-exact.

Proposition 5.2. The Hamiltonian, H, and momentum, P, are Q_A-exact and therefore Q_A-closed.

Solution. It is clear to see H is Q_A-exact by a simple rewriting of equation 5.3a,

$$\begin{aligned} H &= \frac{1}{2}\{\bar{Q}_+ + Q_-, Q_+ - \bar{Q}_-\}, \\ &= \frac{1}{2}\{Q_A, Q_+ - \bar{Q}_-\}. \end{aligned} \tag{5.13}$$

Being exact implies closed as,

$$\begin{aligned} Q_A H &= \{Q_A, H\} \\ &= \frac{1}{2}\{Q_A, \{Q_A, Q_+ - \bar{Q}_-\}\} \\ &= -\frac{1}{2}(\{Q_+ - \bar{Q}_-, \{Q_A, Q_A\}\} + \{Q_A, \{Q_+ - \bar{Q}_-, Q_A\}) \\ &= -\{Q_A, H\} = -Q_A H. \end{aligned} \tag{5.14}$$

Which of course can only be true if the expression is identically zero. Hence, $\{Q_A, H\} = 0$ and then H is Q_A-closed. Where we have used the Jacobi identity in getting to the second to last line in equation 5.13.

The same is true of P. Following the same exact steps we have,

$$\begin{aligned} P &= \frac{1}{2}\{\bar{Q}_+ + Q_-, Q_+ + \bar{Q}_-\} \\ &= \frac{1}{2}\{Q_A, Q_+ + \bar{Q}_-\} \\ \implies Q_A P &= \{Q_A, P\} \\ &= \frac{1}{2}\{Q_A, \{Q_A, Q_+ + \bar{Q}_-\}\} \\ &= -\frac{1}{2}(\{Q_+ + \bar{Q}_-, \{Q_A, Q_A\}\} + \{Q_A, \{Q_+ + \bar{Q}_-, Q_A\}) \\ &= -\{Q_A, P\} = -Q_A P. \end{aligned} \tag{5.15}$$

Again, we must have then $\{Q_A, P\} = 0$ and so P is Q_A-closed as well. □

Propositions 5.1 and 5.2 give us a first taste of a cohomological theory. By defining Q_A we have a potential operator for which we can set up a cohomology. First, however, the fields of the nlσm have new charges after the twist summarized in the following table.

A Twisted Field $U(1)$ Charges

Field	q_V	q_M	q_{M_A}
φ	0	0	0
ψ^i_+	-1	-1	-2
ψ^i_-	-1	1	0
$\bar{\psi}^{\bar{i}}_+$	1	-1	0
$\bar{\psi}^{\bar{i}}_-$	1	1	2

Removing the Lorentz charge untwists the bundles, but now the other bundles have an *extra twist.* The fields now live in the following bundles.

$$\psi_+^i \in S_+ \otimes \varphi^*\big(T^{(1,0)}\big) \tag{5.16a}$$
$$\psi_-^i \mapsto \chi^i \in \varphi^*\big(T^{(1,0)}\big) \tag{5.16b}$$
$$\bar{\psi}_+^{\bar{i}} \mapsto \bar{\chi}^{\bar{i}} \in \varphi^*\big(T^{(0,1)}\big) \tag{5.16c}$$
$$\bar{\psi}_-^{\bar{i}} \in S_- \otimes \varphi^*\big(T^{(0,1)}\big) \tag{5.16d}$$

Where we have renamed the untwisted[1] ψ fields to help distinguish them as new scalar fields. It is important to note that we have not changed the fields statistics only its Lorentz representation. The component fields' transformations from equation 5.4a-f reduce under a Q_A transformation, with $\epsilon_+ = \bar{\epsilon}_- = 1$, as

$$\delta\varphi^i = \chi^i, \tag{5.17a}$$
$$\delta\bar{\varphi}^{\bar{i}} = \bar{\chi}^{\bar{i}}, \tag{5.17b}$$
$$\delta\psi_+^i = 2i\partial_+\varphi^i + \Gamma^i_{jk}\psi_+^j\chi^k \tag{5.17c}$$
$$\delta\bar{\chi}^{\bar{i}} = 0 \tag{5.17d}$$
$$\delta\chi^i = 0 \tag{5.17e}$$
$$\delta\bar{\psi}_-^{\bar{i}} = 2i\partial_-\bar{\varphi}^{\bar{i}} + \Gamma^{\bar{i}}_{\bar{j}\bar{k}}\bar{\psi}_-^{\bar{j}}\bar{\chi}^{\bar{k}}. \tag{5.17f}$$

Proposition 5.3. The A-Model is a topological theory.

Solution. In order to show the theory is topological we will take the route of setting up a cohomological field theory. Proposition 5.1 satisfies axiom (CoFT1) and proposition 5.13 may lead us to think that axiom (CoFT2) is also satisfied. We have only shown that two observables are Q_A-closed and to satisfy (CoFT2) we need to show *all* observable are closed under Q_A. A priori we have no way to account for all the other observables[2] let alone finding a symmetric ground state and trying to find a Q_A-exact energy-momentum tensor. We can circumvent these potential issues by considering the Lagrangian.

Rewriting the nlσm (equation 4.19) with the newly twisted fields gives us,

$$\mathcal{L}_{A-\text{twist}} = -g_{i\bar{j}}\partial^\alpha\varphi^i\partial_\alpha\bar{\varphi}^{\bar{j}} - 2ig_{i\bar{j}}\bar{\psi}_-^{\bar{j}}\mathcal{D}_+\chi^i - 2ig_{i\bar{j}}\bar{\chi}^{\bar{j}}\mathcal{D}_-\psi_+^i - R_{i\bar{j}k\bar{l}}\psi_+^i\chi^k\bar{\chi}^{\bar{j}}\bar{\psi}_-^{\bar{l}}. \tag{5.18}$$

However, under the integral of the action we can do some integration by parts (assuming the fields die off at infinity) and come to the conclusion

$$\mathcal{L}_{A-\text{twist}} = -2\bigg(g_{i\bar{j}}\partial_+\varphi^i\partial_-\bar{\varphi}^{\bar{j}} + g_{i\bar{j}}\partial_-\varphi^i\partial_+\bar{\varphi}^{\bar{j}} + ig_{i\bar{j}}\bar{\psi}_-^{\bar{j}}\mathcal{D}_+\chi^i + ig_{i\bar{j}}\psi_+^i\mathcal{D}_-\bar{\chi}^{\bar{j}} + \frac{1}{2}R_{i\bar{j}k\bar{l}}\psi_+^i\chi^k\bar{\chi}^{\bar{j}}\bar{\psi}_-^{\bar{l}}\bigg). \tag{5.19}$$

In this form, the lagrangian is not very useful. However, using the field variations, equations 5.17a-f, and a specific potential function [20],

$$V = g_{i\bar{j}}(\psi_+^i\partial_-\varphi^j + \partial_+\bar{\varphi}^{\bar{i}}\psi_-^j) \tag{5.20}$$

[1] The fields who now have trivial Lorentz charge were in a twisted bundle but no live in the trivial bundle and so we say untwisted.

[2] We actually will be able to give an account of the general physical observable, but we reserve that for the next section.

we find the lagrangian can be written as

$$\mathcal{L}_{A-\text{twist}} = -it\{Q_A, V\} + 2tg_{i\bar{j}}(\partial_+\varphi^i\partial_-\bar{\varphi}^{\bar{j}} - \partial_-\varphi^i\partial_+\bar{\varphi}^{\bar{j}}). \tag{5.21}$$

We have added a coupling constant, t, and recognize the second term as the Kähler form pulled back to the string worldsheet, $\varphi^*(\omega)$. Equation 5.21, is what we want to see, almost. Our aim is to be able to employ proposition 3.3, which we cannot do until we understand the second term in equation 5.21. Let us inspect this term in the action, that is,

$$S_\omega = t\int_\Sigma \varphi^*(\omega) = t\int_{\varphi(\Sigma)} \omega. \tag{5.22}$$

Equation 5.22 only depends on the homology class of the string embedding, $\varphi(\Sigma)$. Let $\beta \in H_2(M)$, we can then denote the integral over the embedding as $\beta \cdot (\ )$, so equation 5.22 becomes,

$$S_\omega = t\beta \cdot \omega. \tag{5.23}$$

In the path integral, then, we just get an extra factor of $e^{-t\beta\cdot\omega}$, which of course is independent of the worldsheet metric.

So, the Lagrangian, $\mathcal{L}_{A-\text{twist}}$, is Q_A-exact plus a term we understand to be (worldsheet) metric independent, giving an action,

$$S_{A-\text{twist}} = -it\left\{Q_A, \int V\right\} + t\beta\cdot\omega, \tag{5.24}$$

and by proposition 3.3, the A-twisted nlσm is cohomological field theory. Naturally, by proposition 3.2, it is therefore a topological field theory. $\square$

This is wonderful, we have now unlocked the door to access all the tools the topological theory provides us. Let us take a peek and see what we can glean from our newfound CoFT. Notionally, we will refer to the strictly Q_A-exact pieces of $\mathcal{L}_{A-\text{twist}}$ and $S_{A-\text{twist}}$ as $\mathcal{L}_A$ and S_A respectively.

5.2.1 Correlation Functions in the A-Model

Now we really pull on the connection between the field theory and the differential geometry. To investigate the correlation functions we need to give an account of the physical operators, i.e. operators $\mathcal{O}$ such that $\{Q_A, \mathcal{O}\} = 0$. We start with the identification rule that

$$\chi^i \leftrightarrow \mathrm{d}\varphi^i \equiv \mathrm{d}z^i, \quad \bar{\chi}^{\bar{i}} \leftrightarrow \mathrm{d}\bar{\varphi}^{\bar{i}} \equiv \mathrm{d}\bar{z}^{\bar{i}}. \tag{5.25}$$

Recalling that $\delta\mathcal{O} = [Q_A, \mathcal{O}\}$, we can rephrase equations 5.17a-f,

$$[Q_A, \varphi^i] = -\chi^i, \tag{5.26a}$$

$$[Q_A, \bar{\varphi}^{\bar{i}}] = -\bar{\chi}^{\bar{i}}, \tag{5.26b}$$

$$\{Q_A, \chi^i\} = 0, \tag{5.26c}$$

$$\{Q_A, \bar{\chi}^{\bar{i}}\} = 0. \tag{5.26d}$$

So, given some physical operator, $\mathcal{O}_\alpha$, we have

$$[Q_A, \mathcal{O}_\alpha\} = -\mathcal{O}_{\mathrm{d}\alpha}, \tag{5.27}$$

that is Q_A takes on the role of the exterior derivative (the De Rahm cohomology operator). We can read off then $\bar{Q}_+ \leftrightarrow \bar{\partial}$, and $Q_- \leftrightarrow \partial$, the Dolbeaut cohomology operators. It is clear then how we can construct a general physical operator, we have the (p,q)-form,

$$\mathcal{O}_\alpha = \alpha_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q}(\varphi)\chi^{i_1}\cdots\chi^{i_p}\bar{\chi}^{\bar{j}_1}\cdots\bar{\chi}^{\bar{j}_q}. \tag{5.28}$$

Any factors with the ψ fields (1-forms) or derivatives of φ and χ means tacking on worldsheet metrics in order to soak up the $\pm$ indices. Given that we need metric independence, we cannot include these factors.

Taking a generic correlation function with our physical operators as given by equation 5.28 we consider,

$$\begin{aligned}\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle &= \int \mathcal{O}_1 \cdots \mathcal{O}_n e^{-S_{A-\mathrm{twist}}} D\varphi D\bar{\varphi} D\chi D\bar{\chi} D\psi_+ D\bar{\psi}_- \\ &= e^{-t\beta\cdot\omega} \int \mathcal{O}_1 \cdots \mathcal{O}_n e^{-S_A} D\varphi D\bar{\varphi} D\chi D\bar{\chi} D\psi_+ D\bar{\psi}_-.\end{aligned} \tag{5.29}$$

Remember the field φ provides the worldsheet embedding, $\varphi : \Sigma \hookrightarrow M$, and by having φ in the path integral we are then considering *all* possible string embeddings into the Kähler target space. Recall further that the embedding was only dependent on the homology class, $\varphi_*[\Sigma] = \beta \in H_2(M)$ so the correlator becomes a sum over the possible $\beta \in H_2(M)$ [9],

$$\begin{aligned}\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle &= \sum_{\beta \in H_2(M)} \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_\beta \\ &= \sum_\beta \int_\beta \mathcal{O}_1 \cdots \mathcal{O}_n e^{-S_{A-\mathrm{twist}}} D\varphi D\bar{\varphi} D\chi D\bar{\chi} D\psi_+ D\bar{\psi}_-.\end{aligned} \tag{5.30}$$

Moreover, the path integral should be invariant under a Q_A transformation. By inspection of the transformation properties in equations 5.17a-f, we find $\delta S = 0$ if

$$\partial_-\varphi^i = 0, \text{ and } \partial_+\bar{\varphi}^{\bar{i}} = 0. \tag{5.31}$$

Which of course we recognize as the statement that φ is a holomorphic function. This naturally translates to the string embedding; we are looking at holomorphic string embeddings, $\varphi \in Hol(\Sigma, M)$. Where $Hol(\Sigma, M) = \{\varphi : \Sigma \hookrightarrow M \mid \partial_-\varphi = 0\}$. Let us denote the space of holomorphic string embeddings of degree β as

$$\mathcal{M}_\Sigma(M, \beta) = \big\{\varphi \in Hol(\Sigma, M) \mid \varphi_*[\Sigma] = \beta\big\}, \tag{5.32}$$

where we recall β is also the dimension of the cycle as the embedded string. Equation 5.32 is also referred to as the *moduli space of holomorphic maps*. We also note that the requirement of equation 5.31 means the potential, V, (equation 5.20) vanishes.

Example 5.2. From equation 5.32, if $\beta = 0$ we are talking about worldsheet embeddings as 0-dimensional cycles, i.e. points. This means we are associating points of the target space to worldsheets and the moduli space of maps is just the target space itself, $\mathcal{M}_\Sigma(M, 0) = M$.

Selecting $\beta = 1$ means embedding worldsheets as 1-dimensional loops in the target space as in example A.6. This is probably the most intuitive picture of the embeddings as the dimension is low enough to actually draw out. Picking $\beta = 2$ means we are looking at 2-cycles and so on and so forth for higher degrees β.

One more note on the moduli space. Assuming the moduli space, 5.32, is a smooth manifold we can consider the tangent bundle. In regards to the problem at hand, the tangent bundle being the pullback of the tangent bundle of M to Σ, $\varphi^*(TM)$. We can understand a little bit more about this space using the Grothendieck-Riemann-Roch formula 2.6 to determine that

$$\begin{aligned}\dim(H^0(\varphi^*(TM))) - \dim(H^1(\varphi^*(TM))) &= \int_\Sigma ch(\varphi^*(TM))td(\Sigma) \\ &= m(1-g),\end{aligned} \tag{5.33}$$

where m is the dimension of M and g is the genus of Σ.

Now, we already knew M was a Kähler target space because we are working with the nlσm, but let us restrict ourselves further. Let us put on the further assumption that M is truly Calabi-Yau. This lovely assumption guarantees that the axial R-symmetry will no longer be anomalous by removing the ψ zero modes. By removing the ψ zero modes we are left over with some number, k, of χ zero modes as from equation 4.36. The number of χ zero modes, k, corresponds exactly with the tangent space of $\mathcal{M}_\Sigma(M, \beta)$, and by extension with the dimension of the moduli space itself [20]. From equation 5.33 then we have,

$$k = m(1-g) = \dim\big(\mathcal{M}_\Sigma(M, \beta)\big). \tag{5.34}$$

So, we have found that there are $k = m(1-g)$ different possible ways of holomorphically embedding the string as a β-cycle. So, our path integral over φ is a finite integral.

The above discussion of the moduli space means that the integral in the correlation function in equation 5.30 really becomes an integral over the moduli space,

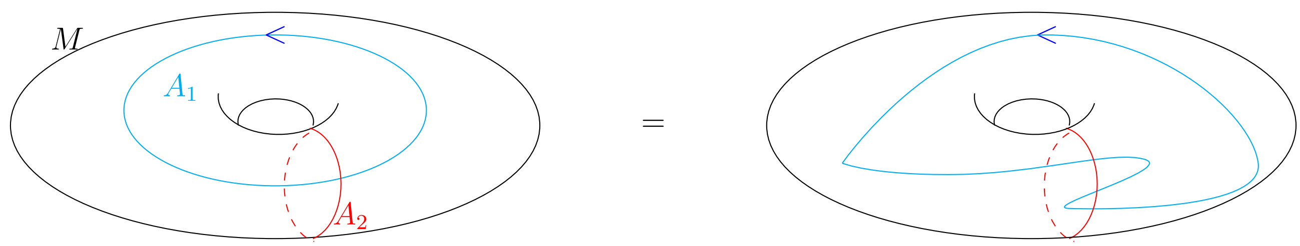
$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_\beta = \int_{\mathcal{M}_\Sigma(M,\beta)} \mathcal{O}_1 \cdots \mathcal{O}_n e^{-S_A} D\varphi D\bar{\varphi} D\chi D\bar{\chi} D\psi_+ D\bar{\psi}_-. \tag{5.35}$$

Of course $\mathcal{O}_1 \cdots \mathcal{O}_n$ must be a top form of the moduli space. That is, equation 5.35 tells us that the sum of the degrees from all the $\mathcal{O}_i$ must be k as from equation 5.34.

Now, to each p-form $\mathcal{O}_i$ we have an associated $(m-p)$-cycle, A_i, its Poincaré dual[3], in M. Then integrals over the forms is interpreted geometrically to be a simple count of the overlap of the cycles accounting for orientation.

[3]See equation A.3.

Example 5.3. We can count how many times the cycles A_1 and A_2 intersect (with sign) by $\int_M \alpha_1 \wedge \alpha_2 = \int_{A_1} \alpha_2 = (-1)^{pq} \int_{A_2} \alpha_1$ where $\alpha_1 \in H^p$ and $\alpha_2 \in H^q$, and A_1, A_2 are the associated Poincaré dual cycles. In the integral over the manifold M we only get contributions from where $\delta(A_1)$ overlaps $\delta(A_2)$ i.e. from $A_1 \cap A_2$. An associated picture is given below in the case where $M \cong T^2$ and A_1, A_2 are two 1-cycles.



This picture demonstrates the fact that the two cyan cycles are in the same homology class as we account for orientation of the cycles leading to the fact that the intersection number of the cyan cycle with red cycle is one.

In this *delta function representation* we only get nonzero results whenever the $\mathcal{O}_i$ hit the corresponding I_i cycle. So, the integral is counting the embeddings where $\varphi(x^i) \in A_i$. For our convenience we define this to be N:

$$N(\beta, A_1, \cdots, A_n) \equiv (\text{the number of } \varphi \in \mathcal{M}_\Sigma(M, \beta) \text{ such that } \varphi(x^i) \in A_i \ \forall\ i). \tag{5.36}$$

So, our general correlation function

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \sum_\beta e^{-t\beta\cdot\omega} N(\beta, A_1, \cdots, A_n). \tag{5.37}$$

Hence, the correlators of the A-twisted model are providing a weighted counting of intersections on the target space weighted by the target space's Kähler form, ω!

Example 5.4. Consider the case from example 5.2, where $\beta = 0$ and we the embedded worldsheet is genus 0, a sphere. The moduli space is the same dimension as the target space, M, and actually is isomorphic to M. So, our general correlator, $\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle$, must be an m-form, assuming M is m-dimensional. So the path integral becomes an integral over the target space M and we have (using the wedge notation to emphasize the form nature of the $\mathcal{O}$'s),

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \int_M \mathcal{O}_1 \wedge \cdots \wedge \mathcal{O}_n = c(A_1 \cap \cdots \cap A_n), \tag{5.38}$$

for some $c \in \mathbb{R}$.

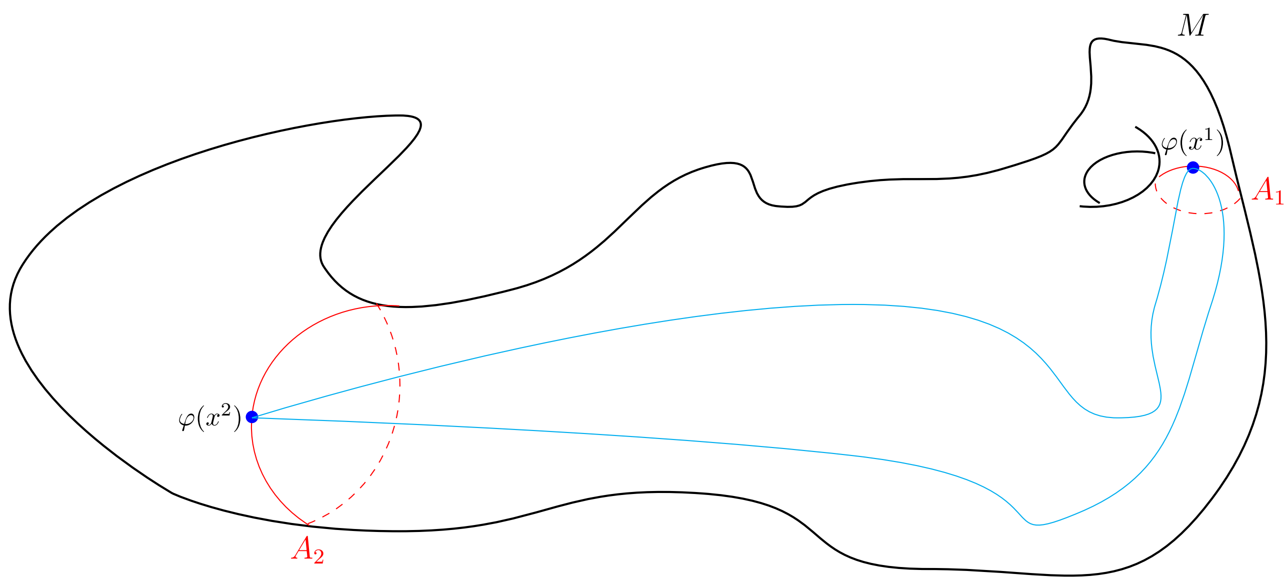
The above example demonstrates, in the rather trivial case with $\beta = g = 0$, the topological theory is just calculating classical intersection numbers. This gives us the physical intuition to interpret equation 5.37

as quantum corrections to intersection numbers! We given another example of a worldsheet of genus 1 below.

Example 5.5. Let us consider the case where the worldsheet is a genus 1, a torus. We can think of this as the first quantum correction to equation 5.38. From equation 5.34, we see the dimension of the moduli space, $\mathcal{M}_\Sigma(M, \beta)$, is 0. We note that a zero dimensional space is one that is composed of isolated points. In this circumstance the points can be counted (think of the natural numbers versus the real numbers which are *countably* infinite and *uncountably* infinite respectively). The correlator here then is just the weighted count of the holomorphic curves in the torus.

Another example this time with an embedded string to give us a mental picture.

Example 5.6. Consider some arbitrary manifold M, here we take M to be some genus 1 object. Further consider two physical operators, $\mathcal{O}_1$ and $\mathcal{O}_2$, with their associated cycles, A_1 and A_2 in M. At some given time we may have the string embedding as pictured.



If the embedded string (path in cyan) connects the two cycles in this manner there will be a nonzero contribution in the path integral. However, as we recall the topological theory is independent of the parameter t. So, we may consider taking the limit $t \to \infty$. In which case the weighting of our count, the $e^{-t\beta\cdot\omega}$ prefactor, goes to zero, leading to a vanishing result. We see this in the illustration above, the cycles A_1 and A_2 actually do not intersect, even though our TQFT says they are. As the string propagates in the $t \to \infty$ limit the contribution will vanish leading to the expected null result.

It vanishes unless we are in the case of example 5.5, $\beta = 0$. Where now, we only get contributions from cycles with 0 volume. That is, we only get contributions from constant embeddings $\varphi : S^2 \to x^i \in M$. So, we see $\varphi(x^i) = x^i \in A_i \ \forall i$. Implying our correlation function is the traditional intersection number, $\langle \mathcal{O}_1(x^1) \dots \mathcal{O}_k(x^k) \rangle_{t\to\infty} = \#(A_1 \cap A_2 \cap \dots \cap A_k)$.

Our A-twisted TQFT provides a new geometric invariant for CY_n which is a *stringy generalization* of intersection number. Let us see how the B-twist compares.

5.3 The B-model

Following the previous section, but this time we chose the axial R-symmetry instead. A brief note about choosing the A or B model should be made here. The vector R-symmetry is never anomalous and our target space would be some general Kähler manifold[4]. The axial R-symmetry is anomalous, however, and in order to have the symmetry survive in our quantum theory the target space needs to be Calabi-Yau. Just to reiterate, picking the A-model means working on a Kähler target space, whereas picking the B-model means working on a Calabi-Yau target space. Now, the new twisted Lorentz generator is,

$$M_B = M - F_A, \tag{5.39}$$

and so our B-twisted Lorentz charge, q_{M_B}, is

$$q_{M_B} = q_M - q_A. \tag{5.40}$$

Then the B-twisted Lorentz generator satisfies the following commutation relations.

$$[M_B, Q_+] = -2Q_+ \tag{5.41a}$$
$$[M_B, Q_-] = 2Q_- \tag{5.41b}$$
$$[M_B, \bar{Q}_+] = 0 \tag{5.41c}$$
$$[M_B, \bar{Q}_-] = 0 \tag{5.41d}$$

and so we pick the operator to be

$$Q_B = \bar{Q}_+ + \bar{Q}_-. \tag{5.42}$$

Just as before we verify it is nilpotent.

Proposition 5.4. The new B-twisted operator, Q_B, is nilpotent.

[4]We actually choose to work on a Calabi-Yau to help simplify things. One could just as well worked in the general Kähler space, but an alternate course of action needs to be taken in order to account for zero modes in the correlation functions.

Solution. Again, this is straight forward to compute,

$$
\begin{aligned}
Q_B^2 &= \bar{Q}_+^2 + \bar{Q}_-^2 + \{\bar{Q}_+, \bar{Q}_-\} \\
&= 0.
\end{aligned}
\tag{5.43}
$$

Hence Q_B is nilpotent. $\square$

From equation 5.3a we can also see the Hamiltonian and momentum operators are also Q_B-exact and therefore Q_B-closed.

Again we summarize the fields twisted and untwisted fields for the B-twist in the following table.

B Twisted Field $U(1)$ Charges

Field	q_A	q_M	q_{M_B}
φ	0	0	0
ψ_+^i	-1	-1	-2
ψ_-^i	1	1	2
$\bar{\psi}_+^{\bar{i}}$	1	-1	0
$\bar{\psi}_-^{\bar{i}}$	-1	1	0

The bundles in which these fields live is as follows.

$$
\begin{aligned}
\psi_+^i &\in S_+ \otimes \varphi^*\big(T^{(1,0)}\big) \\
\psi_-^i &\in S_- \otimes \varphi^*\big(T^{(1,0)}\big) \\
\bar{\psi}_+^{\bar{i}} &\in \varphi^*\big(T^{(0,1)}\big) \\
\bar{\psi}_-^{\bar{i}} &\in \varphi^*\big(T^{(0,1)}\big).
\end{aligned}
\tag{5.44}
$$

In the A-model we conveniently had scalar (1,0)- and (0,1)-forms which lead nicely to the form-physical operator association in equation 5.25. Notice here, however, we have two scalars would be identified as (0,1)-forms. We will see the consequence of this in a moment, first we note the field variations as before.

Wishing to write the variations in the most simplified fashion, we choose to make a relabeling of the scalar fields,

$$
\begin{aligned}
\eta^{\bar{i}} &= -\big(\bar{\psi}_+^{\bar{i}} + \bar{\psi}_-^{\bar{i}}\big), \\
\theta_i &= g_{i\bar{j}}\big(\bar{\psi}_+^{\bar{j}} - \bar{\psi}_-^{\bar{j}}\big).
\end{aligned}
\tag{5.45}
$$

The component field variations in the B-model are given by equation 5.4a with $\epsilon_+ = \epsilon_- = 0$ (and setting

$\bar{\epsilon}_+ = \bar{\epsilon}_- = 1$ for simplicity) we have,

$$\delta\varphi^i = 0, \tag{5.46a}$$
$$\delta\bar{\varphi}^{\bar{i}} = \eta^{\bar{i}}, \tag{5.46b}$$
$$\delta\theta_i = 0, \tag{5.46c}$$
$$\delta\eta^{\bar{i}} = 0 \tag{5.46d}$$
$$\delta\psi^i_- = -2i\partial_-\varphi^i, \tag{5.46e}$$
$$\delta\psi^i_+ = 2i\partial_+\varphi^i \tag{5.46f}$$

Proposition 5.5. The B-model is a topological quantum field theory.

Solution. Just as before we are seeking a way in which we can write the lagrangian in a Q_B-exact way and then employ proposition 3.3. We start by rewriting the nlσm lagrangian in terms of our twisted fields,

$$\mathcal{L}_{B-\text{twist}} = g_{i\bar{j}}\partial^\alpha\varphi^i\partial_\alpha\bar{\varphi}^{\bar{j}} + ig_{i\bar{j}}\eta^{\bar{j}}\big(\mathcal{D}_-\psi^i_+ + \mathcal{D}_+\psi^i_-\big) + i\theta_i\big(\mathcal{D}_-\psi^i_+ - \mathcal{D}_+\psi^i_-\big) + \frac{1}{2}R_{i\bar{j}k}{}^l\psi^i_+\psi^k_-\eta^{\bar{j}}\theta_l. \tag{5.47}$$

Again, we need to find a potential function. For the B-model we take,

$$V = g_{i\bar{j}}\big(\psi^i_+\partial_-\bar{\varphi}^{\bar{j}} + \psi^i_-\partial_+\bar{\varphi}^{\bar{j}}\big). \tag{5.48}$$

So, the B-twisted lagrangian becomes (adding in the coupling constant $-t$),

$$\mathcal{L}_{B-\text{twist}} = -it\{Q_B, V\} - t\left(i\theta_i\big(\mathcal{D}_-\psi^i_+ - \mathcal{D}_+\psi^i_-\big) + \frac{1}{2}R_{i\bar{j}k}{}^l\psi^i_+\psi^k_-\eta^{\bar{j}}\theta_l\right). \tag{5.49}$$

This is great! The first term is exactly as we need and the second term is anti-symmetric in $\pm$ indices implying it is a (1,1)-form which when integrated over will be independent of the worldsheet metric [6]. Therefore, the B-model is cohomological and therefore topological. □

5.3.1 Correlation Functions in the B-Model

Following in the same manner as the A-model, we make the identification

$$\eta^{\bar{i}} \leftrightarrow \mathrm{d}\bar{\varphi}^{\bar{i}} \equiv \mathrm{d}\bar{z}^{\bar{i}}, \quad \theta_i \leftrightarrow \frac{\partial}{\partial\varphi^i} \equiv \frac{\partial}{\partial z^i}. \tag{5.50}$$

rewriting the field variations we have,

$$[Q_B, \varphi^i] = 0 \tag{5.51a}$$
$$[Q_B, \bar{\varphi}^{\bar{i}}] = \eta^{\bar{i}} \tag{5.51b}$$
$$\{Q_B, \theta_i\} = 0 \tag{5.51c}$$
$$\{Q_B, \eta^{\bar{i}}\} = 0. \tag{5.51d}$$

So we write a general physical operator in the B-model as,

$$\mathcal{O}_\beta = \beta_{\bar{i}_1\cdots\bar{i}_p}{}^{j_1\cdots j_q}(\varphi,\bar{\varphi})\eta^{\bar{i}_1}\cdots\eta^{\bar{i}_p}\theta_{j_1}\cdots\theta_{j_q}, \tag{5.52}$$

which is not quite a p-form, but rather a p-form that also lives in the $\bigwedge^q T^{(1,0)}$ bundle. There are a few observations to be made here.

1. Saying $\mathcal{O}_\beta$ is a p-form is slightly misleading in this case as in reality it is p anti-holomorphic indices. In this way we recognize the B-twisted operator, Q_B, with the Dolbeault operator,

$$[Q_B, \mathcal{O}_\beta\} = \mathcal{O}_{\bar{\partial}\beta}. \tag{5.53}$$

   So the Q_B-cohomology is identified as the $\bar{\partial}$-cohomology.

2. We see a key distinction between the A- and B-model come alive in how they depend on the target space. The decent equations of the B-model tell us there is a dependence on the target space's complex structure. We did not have that anywhere in the A-model[5].

Just as before, we should take the action to be Q_B-invariant. This forces

$$\partial_\pm\varphi^i = 0, \quad \partial_\pm\bar{\varphi}^{\bar{i}} = 0 \tag{5.54}$$

to all be true. This is to say that φ is a constant map; a constant worldsheet embedding! As in the A-model, the constant embeddings mean the moduli space is really just the target space, M, itself. The path integral then appears to be an integral over the target space (again say dimension m) of a $(0,p)$-form that also happens to live in the q^{th} power holomorphic tangent bundle, $\bigwedge^q T^{(1,0)}$. Well that is not good, and if we want to fix this issue we need to do so in a strictly topological way. That is, we want to preserve the topological nature of our theory but still be able to compute correlation functions. Following the prescription of [6] and [9], the only way we can solve this is by absorbing the holomorphic indices with the Calabi-Yau form, Ω, then tacking on another Calabi-Yau form to give us a (m,m)-form to properly integrate[6]. In components we map,

$$\beta_{\bar{i}_1\cdots\bar{i}_m}{}^{j_1\cdots j_m} \mapsto \beta_{\bar{i}_1\cdots\bar{i}_m}{}^{j_1\cdots j_m}\Omega_{j_1\cdots j_m}\Omega_{k_1\cdots k_m}. \tag{5.55}$$

So, the correlation functions of the B-model are just integrals of forms over the target space.

$$\begin{aligned}\langle\mathcal{O}_{\beta_1}\cdots\mathcal{O}_{\beta_n}\rangle &= \int \mathcal{O}_{\beta_1}\cdots\mathcal{O}_{\beta_n}e^{-S_{B-\text{twist}}}D\varphi D\bar{\varphi}D\eta D\theta \\ &= \int_M \langle(\mathcal{O}_{\beta_1}\wedge\cdots\wedge\mathcal{O}_{\beta_n}),\Omega\rangle\wedge\Omega\end{aligned} \tag{5.56}$$

where $\langle A, B\rangle$ denotes index contraction of A and B.

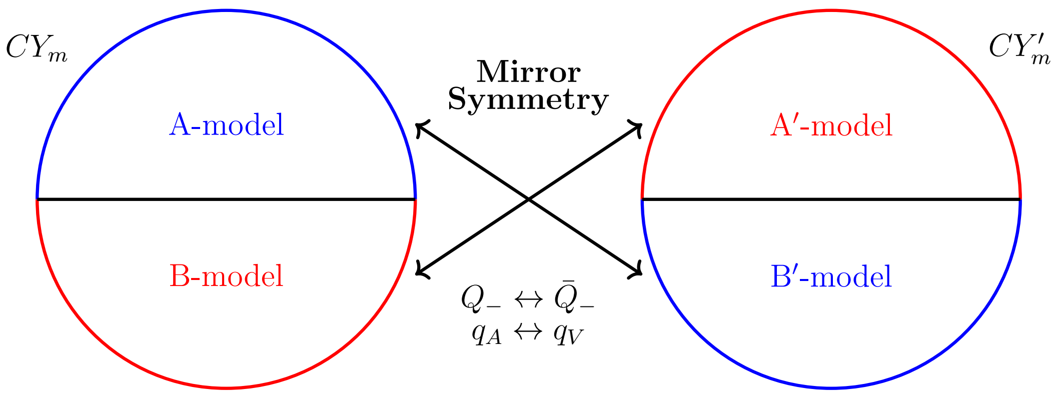
[5]Recall the A-model led to a dependence on the target space's Kähler form.

[6]Remember we can only integrate over top forms so we should only consider correlators in which the product of $\mathcal{O}$'s is a $(0,m)$-form. Vanishing anomalies requires that the holomorphic indices sum to m as well

5.4 Mirror symmetry

Notice in equation 5.3a that Q_- and $\bar{Q}_-$ can be exchanged and the anti-commutator still holds. Equations 5.11 and 5.42 are also interchanged, and if we also swap q_A and q_V, we still have the $\mathcal{N} = (2,2)$ supersymmetry with the same A and B twisted models. Say we had two $\mathcal{N} = (2,2)$ supersymmetric quantum field theories[7] (labeled (i) and (ii)), if there is an isomorphism between them in which $Q_-^{(i)} \mapsto \bar{Q}_-^{(ii)}$ and $q_A^{(i)} \mapsto q_V^{(ii)}$, then then theory (i) is mirror to theory (ii) [9].

We can take a step back and look at the broader context in which these theories were defined. We put the nlσm on a Calabi-Yau target space. Remember, way back in section 2.3.4, Calabi-Yau manifolds come in *mirror* pairs. So, installing a topological theory on a CY_m means we have done something to its mirror, CY_m' as well. What this means is that the A-model on CY_m corresponds to the B-model on its mirror CY_m'. This is truly amazing. What was a tricky computation in finding the intersection numbers in the A-model by integrating over the moduli space has now become a straight forward integral over the Calabi-Yau in the B-model on its mirror pair [6, 9]!



We have come a long way. Starting from the $\mathcal{N} = (2,2)$ supersymmetric non-linear sigma model on the string worldsheet, we have found, naturally leads to mirror symmetry of Calabi-Yau manifolds. Whats more, we found, this theory gives a way in which we can calculate geometric (topological) invariants on target spaces using quantum field theory.

[7] Meaning no anomalies, see section 4.4.

Chapter 6

Final Remarks

Having concluded the main results of this project, we move on to make several final remarks. The aim of this chapter is not to discover new results, but rather to summarize the findings of the previous chapters.

We have truly come a very long way from our journeys start. From our deep dive into differential geometry, we discovered the Calabi-Yau's hiding right under our noses while getting our first glimpse of mirror symmetry. Then on to a lengthy discussion of topological quantum field theories, laying the groundwork for future results. Then, we want to see one of these TQFTs in action, so we introduced the nlσm. In this model we find the symmetry on the worldsheet means Kähler target space, and by canceling R-symmetry anomalies the target space becomes CY_m. We also find in this topological theory, geometric invariants, that is intersection number, are calculated in the A-model. By accessing the mirror symmetry of the Calabi-Yau target space, the A- and B-models are related on the mirror pair. Before ending, we give a final remark on the worldsheet embeddings in the A-model.

6.1 Tying up Loose Ends

There is a subtlety we have waited to mention about our worldsheet embeddings in the A-model. Recall from equation 5.34 the dimension of the moduli space of holomorphic maps was dependent on the worldsheet genus as $m(1-g)$. Well, as may have already been guessed there is an issue if $g > 1$. If $g \leq 1$ we are in the situations described at length in the previous chapter. Finding $g > 1$ means a negative dimension for the moduli space, which can be interpreted as a statement about the lack of holomorphic worldsheet embeddings. The lack of holomorphic maps at higher genus can be remedied, however, by transitioning from the traditional field theory to string theory. By including worldsheet geometries in our topological models higher order genus surfaces can be included. This is done most easily on a certain target spaces.

Recall back to Chapter 2, to a lengthy discussion about how compactification of the extra six dimensions in string theory leads us naturally to Calabi-Yau manifolds, specifically CY_3. We just cannot seem to get enough of them as CY_3 appears yet again in the topological string theory. Without going through all the details, anomaly cancellation occurs when the Calabi-Yau target space is specifically (you guessed it) CY_3 [6]. Meaning that on CY_3 correlation functions at any genus is non-vanishing.

By transitioning to string theory, one will find that on the Calabi-Yau target space, the type IIA and IIB string theories correspond to the A- and B-models respectively[1] [9]. So, through mirror symmetry the type IIA string theory on CY_m corresponds to the type IIB theory on its mirror partner CY'_m.

6.2 If You Give a Mouse a Cookie

Our journey through topological quantum field theories and mirror symmetry has only truly begun. We have simply laid the framework for topological string theory and real calculations to be done. The natural next step, as briefly discussed above, is to formally turn the TQFT into topological string theory. From here, one may be led to study the topological nature of D-branes, matrix models, and even black holes [6, 9].

The soul focus of this work was on the A- and B-twist of the non-linear sigma model. Of course there are other twistable theories out there. To name one, there are Landau-Ginzburg models which arise by considering holomorphic superpotential functions. Compared to our case where the theory arose from D-terms in the lagrangian. The procedure for arriving at the topological nlσm here will not change in looking at different models, although end results will vary.

On the mathematical side, we have done something incredibly exciting. We have turned the tables; finding geometric invariants can now be done through the tools and techniques of quantum field theory. Rather than the usual scenario in which physics is done by the tools of mathematics, mathematics can now be done by the tools of physics! Of course the spaces in question have to be able to support a quantum field theory which is a lot to ask. Nonetheless, this still provides a magnificent bridge between the worlds of math and physics.

[1]This is how the A- and B-models were named.

Appendix A

Preliminaries

A.1 Topology

Definition A.1. A topology over a set X with another set $\mathcal{T}$, where $\mathcal{T}$ is a collection of subsets, $U \subseteq X$ ($\{U_i | U_i \subseteq X\}$) s.t.

T1. $\mathcal{T}$ has the empty set and the whole set X: $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,

T2. A finite intersection of elements of $\mathcal{T}$ is also in $\mathcal{T}$: $\bigcap_{i=1}^{n} U_i \subset \mathcal{T}$, and

T3. an arbitrary collection of elements of $\mathcal{T}$ is also in $\mathcal{T}$: $\bigcup_i U_i \subset \mathcal{T}$

X together with $\mathcal{T}$ makes X a topological space, $(X, \mathcal{T})$. The U_i are called open sets and if $x \in U_x \subset \mathcal{T}$ then U_x is a neighborhood of $x \in X$.

Example A.1. Let $X = \{\text{dog, cat, goldfish}\}$ and the topology $\mathcal{T}$ over X be $\mathcal{T} = \{\emptyset, \{\text{cat, goldfish}\}, X\}$. Then we check topology axioms:

T1) By inspection $\emptyset, X \in \mathcal{T}$,

T2) As this set only has three elements we can easily verify all possible intersections (and unions). Any intersection with the empty set is also empty (which is an element of the topology), so we just need to check: $\{\text{cat, goldfish}\} \cap X = \{\text{cat, goldfish}\} \in \mathcal{T}$, and

T2) Any union containing X will be X (as other elements of $\mathcal{T}$ are subsets of X) and then $\emptyset \cup \{\text{cat, goldfish}\} = \{\text{cat, goldfish}\} \in \mathcal{T}$.

Thus the set X with set $\mathcal{T}$ forms a topological space $(X, \mathcal{T})$ with topology $\mathcal{T}$ over X.

Definition A.2. A topological space, $(M, \mathcal{A})$ is said to be Hausdorff if for every two distinct points, $p, q \in M$, $\exists$ open neighborhoods $U_p, U_q \in \mathcal{A}$ s.t. $U_p \cap U_q = \emptyset$ (U_p and U_q are said to be disjoint).

The definition of a Hausdorff topological space makes concrete the very familiar idea that two points are separated or far off this is a fundamental idea for everything we deal with in our universe. It is worth noting that a Hausdorff space means limits of sequences are unique and compact subsets of a Hausdorff space are also closed, another big result. There is a lovely picture to demonstrate this definition:

Example A.2. Referring back to example A.1, the topological space $(X, \mathcal{T})$ is not Hausdorff. To see this, if we take the elements dog and cat, can you find an open set that contains dog and an open set that contains cat such that the open sets are disjoint?

One of the key ideas about topology is understanding whether two topological spaces are the same or not. We can determine two spaces as the same if one can be continuously transformed into the other and back again.

Definition A.3. Let $(X, \mathcal{T})$ and $(Y, \mathcal{O})$ be two topological spaces with $U \in \mathcal{T}$ and $V \in \mathcal{O}$, and $f : X \to Y$. If $\text{preim}_f(\{O\}) = U \in \mathcal{T}$ for some $U \in \mathcal{T}$ then f is continuous. If f is a bijection between X and Y it has an inverse. If f^{-1} is also continuous then f is a homeomorphism.

This is making precise the idea of being able to change one space into another, like the coffee cup into the doughnut and back again. In the way if the two spaces have a homeomorphism between them they are really the same space.

A refresher of compactness. In our everyday language we say things are compact meaning the object is 'dense' or 'small' this carries over to the rigorous math definition in the loose sense that we want to have a notion of small or large. We now make this definition precise.

Definition A.4. A (open) cover of a topological space $(X, \mathcal{T})$, is collection of open sets, U_α of $\mathcal{T}$ such that $X \subseteq \bigcup_{\alpha \in C} U_\alpha$ where $C \subseteq \mathcal{T}$.

Definition A.5. A topological space X is compact if for every cover, C, of X $\exists$ a finite sub-cover, $S \subset C$ of X (that covers X).

Example A.3. Here we remind the reader of how the stereographic projection of the 2-sphere to $\mathbb{R}^2$ is derived. Start by defining two charts: $\pi_N : S^2 \to \mathbb{R}^2$ and $\pi_S : S^2 \to \mathbb{R}^2$ where N is the north pole chart in which a line is drawn from the north pole, $Np = \langle 0, 0, 1\rangle \in S^2$, of the sphere to any (all) points in $\mathbb{R}^2$. Where the line drawn from the north pole intersects the 2-sphere is the point on the sphere that corresponds to the point where the line intersects the real plane. Likewise for the south pole chart with $Sp = \langle 0, 0, -1\rangle \in S^2$. Then define the line between the north pole (and south pole) and the intersection point on the sphere as

$$\begin{aligned} L &= \lambda\big(\langle 0,0,1\rangle - \langle x,y,z\rangle\big) - \langle 0,0,1\rangle \\ &= \langle -\lambda x, -\lambda y, \lambda - \lambda z - 1\rangle, \end{aligned}$$

where we multiply by lambda to extend the line so it hits the real plane. When the line hits the plane, we have, for $a, b \in \mathbb{R}$,

$$\begin{gathered} -\lambda x = a, \quad -\lambda y = b, \quad \lambda - \lambda z - 1 = 0, \\ \text{Intersect plane when } \lambda = \frac{1}{1-z}, \\ \text{so } a = \frac{-x}{1-z}, \quad b = \frac{-y}{1-z}, \\ \text{and we have } \pi_N : \langle x,y,z\rangle \mapsto \left\langle \frac{-x}{1-z}, \frac{-y}{1-z} \right\rangle \in \mathbb{R}^2 \end{gathered}$$

for the north pole chart. Likewise for the south pole chart (with $c, d \in \mathbb{R}$) but this time $\lambda = \frac{1}{1+z}$ and then

$$\begin{gathered} c = \frac{x}{1+z}, \quad d = \frac{y}{1+z}, \\ \text{and we have } \pi_S : \langle x,y,z\rangle \mapsto \left\langle \frac{x}{1+z}, \frac{y}{1+z} \right\rangle \in \mathbb{R}^2. \end{gathered}$$

A.2 Complex Analysis

Definition A.6. A function $f : \mathbb{C}^n \to \mathbb{C}^n$ is holomorphic if $\frac{df}{d\bar{z}_i} = 0 \ \forall i \in \{1, \ldots, n\}$[1]. f may also be called analytic.

That is to say f is holomorphic if it does not depend on the conjugate variable $\bar{z}$.

Definition A.7. Let $f : M \to N$ be a holomorphic map. If $\exists\, f^{-1} : N \to M$ such that f^{-1} is holomorphic then f is a biholomorphic function.

Theorem A.1 (Maximum Modulus)**.** *Let $f(z)$ be a holomorphic function in some subset D bounded by some contour C of the complex plane. If $|f(z)| \leq M$ on C, then $|f(z)| < M$ on D unless $f(z)$ is constant.*

following from[21].

[1]This is equivalent to saying f satisfies the Cauchy-Riemann equations

A.3 Rings and Fields

Definition A.8. A set R with two binary operations $(\clubsuit, \heartsuit)$, $(R, \clubsuit, \heartsuit)$ is a ring if $(R, \clubsuit)$ is an *abelian group*,

R1) $\exists\ e \in R$ s.t. $\forall\ r \in R\ e\clubsuit r = r\clubsuit e = r$ (identity),

R2) $\forall\ r \in R\ \exists\ s \in R$ s.t. $r\clubsuit s = e = s\clubsuit r$ (inverse)[2],

R3) $\forall\ r, s, t \in R\ (r\clubsuit s)\clubsuit t = r\clubsuit(s\clubsuit t)$ (associative), and

R4) $\forall\ r, s \in R\ r\clubsuit s = s\clubsuit r$ (abelian),

furthermore $(R, \heartsuit)$ is a *monoid*,

R5) $\exists\ \tilde{e} \in R$ s.t. $\forall r \in R\ \tilde{e}\heartsuit r = r\heartsuit\tilde{e} = r$ (identity), and

R6) $\forall\ p, q, r \in R\ (p\heartsuit q)\heartsuit r = p\heartsuit(q\heartsuit r)$ (associative),

and Lastly there is a distributive property,

R7) $\forall\ p, q, r \in R\ p\heartsuit(q\clubsuit r) = p\heartsuit q\clubsuit p\heartsuit r$ (left distributive), and

R8) $\forall\ p, q, r \in R\ (q\clubsuit r)\heartsuit p = q\heartsuit p\clubsuit r\heartsuit p$ (right distributive).

Of course there is also the zeroth axiom in which the ring is closed under both operations. We say $\clubsuit$ is an 'additive' operation (addition) and $\heartsuit$ is a 'multiplicative' operation (multiplication).

If the ring R commutes under the multiplication then we call it a commutative ring.

Definition A.9. A field, $(F, +, \cdot)$, is a commutative ring $(a \cdot b = b \cdot a)$ with the additional property that

$$\forall\ a \in F\ \exists\ b \in F,\ b \neq 0, \text{ s.t. } a \cdot b = \tilde{e} = b \cdot a.$$

Here $\tilde{e}$ the identity under $\cdot$ is different from e the identity under $+$.

This is to say a field is a commutative ring in which all nonzero elements of a field have a multiplicative inverse. A Field allows us to define what subtraction and division mean.

[2] s can also written as $-r$

Example A.4. $\mathbb{Z}, \mathbb{C}, \mathbb{R}$, under standard everyday addition and multiplication.

Definition A.10. Let k be a field. The ring of polynomials, $k[x]$, (under addition and multiplication) is

$$k[x] = \left\{f = \sum_{i=1}^{d} a_i x^i \mid a_i \in k\right\}.$$

The ring of polynomials in n variables is

$$k[x_1, \ldots, x_n] = \left\{f = \sum_{i_1,\cdots,i_n}^{d} a_{i_1,\ldots,i_n} x^{i_1} \cdots x^{i_n} \mid a_{i_1,\ldots,i_n} \in k\right\}.$$

Example A.5. Take $f = (\mathbb{R}, +, \cdot)$ and let polynomial ring $\mathbb{R}[x] \supset S = \{x^2 - 4, x^3 - x, 5\}$.

A.4 Homology and Cohomology

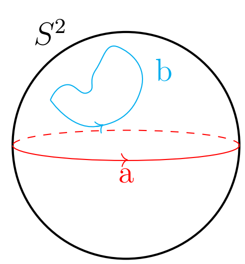
Homology gives a rigorous way to find and discuss holes in a manifold so as to distinguish the manifolds. In much the same way as we define cohomology, we can define homology with some nilpotent operation. To start we first need to consider p-dimensional paths on manifolds.

Definition A.11. Let S be an p-dimensional submanifold of m-dimensional M. We call S a p-chain if it lives in a vector space of other p-dimensional submanifolds on M. If the boundary of the p-chain, S, is zero, $\partial S = 0$, then S is a p-cycle.

Note that the boundary is one dimension less, so the *boundary map*, ∂ maps p-chains to $(p-1)$-chains. Of course, the boundary of the boundary is 0, $\partial V = 0$, if V is already a boundary, $V = \partial W$. Also, p-cycles are given an orientation.

Definition A.12. The p-dimensional submanifold S is a p-boundary if S is already a boundary of some $(p+1)$-chain, T. That is $S = \partial T$.

Example A.6. Some p-cycles, a and b, on the sphere with $p = 1$.



The cycles in the above example tell us that the sphere encloses a 2-dimensional hole.

Just as with cohomology we can set up an equivalence of p-cycles. That is two p-cycles are in the same equivalence class if they differ by a p-boundary. Then the Homology group is the set of all the equivalence classes or simply all those p-cycles that are not p-boundaries,

$$H_p(M) = \frac{\{S|\partial S = 0\}}{\{S|S = \partial T\}}. \tag{A.1}$$

Homology is naturally dual to cohomology by integrating p-forms over p-cycles.

$$\int_S \alpha = r \in \mathbb{R} \quad \forall\, S \in H_p(M),\ \alpha \in H^p(M) \tag{A.2}$$

Poincaré Duality: Every De Rahm cohomology class $\alpha \in H^P$ is dual to some homology class $A \in H_{m-p}$ defined via:

$$\int_M \alpha \wedge \beta = \int_A \beta \quad \forall\, \beta \in H^{m-p}. \tag{A.3}$$

To which we interpret this as a delta-like function. That is to say we only get contributions from the overlap of α and β. The duality between the homology and cohomology is such that picking any $\beta \in H^p M$ uniquely determines a $S \in H_p M$ and vice versa.

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